Heilbronn, 30.9.2025 Prof. Dr. V. Stahl WS 25/26

Homework for Digital Signal Processing with Solutions

Sheet 1

Exercise 1. The convolution $f * g$ of two functions f, g ist defined by

$$(f * g)(t) \quad = \quad \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

Let $f \in \mathbb{R} \to \mathbb{R}$ und $c \in \mathbb{R}$. The function cf ist defined by

$$cf \in \mathbb{R} \to \mathbb{R}, \quad (cf)(t) = cf(t).$$

Show that

$$(cf) * g \quad = \quad c(f * g).$$

Solution for Exercise 1.

$$\begin{aligned} (cf) * g &= \int_{-\infty}^{\infty} (cf)(\tau)g(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} cf(\tau)g(t-\tau)d\tau \\ &= c\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \\ &= c(f * g). \end{aligned}$$

Exercise 2. The delay of a signal f by $\hat{t}$ is indicated by an index $\hat{t}$, i.e.

$$f_{\hat{t}}(t) \quad = \quad f(t-\hat{t}).$$

Show that

$$f_{\hat{t}} * g \quad = \quad (f * g)_{\hat{t}}.$$

Solution for Exercise 2.

$$\begin{aligned} (f_{\hat{t}} * g)(t) &= \int_{-\infty}^{\infty} f_{\hat{t}}(\tau)g(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau-\hat{t})g(t-\tau)d\tau. \end{aligned}$$

Substitution

$$\mu = \tau - \hat{t}, \quad \frac{d\mu}{d\tau} = 1, \quad d\tau = d\mu.$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\tau - \hat{t}) g(t-\tau) d\tau &= \int_{-\infty}^{\infty} f(\mu) g(t - (\mu + \hat{t})) d\mu \\
&= \int_{-\infty}^{\infty} f(\mu) g(t - \hat{t} - \mu) d\mu \\
&= \int_{-\infty}^{\infty} f(\tau) g(t - \hat{t} - \tau) d\tau \\
&= (f * g)(t - \hat{t}) \\
&= (f * g)_{\hat{t}}(t).
\end{aligned}$$

Exercise 3. Let g be a T-periodic function. Show that $f*g$ is also a T-periodic function for any function f.

Solution for Exercise 3. Let g be a T-periodic function, i.e.

$$g(t+T) = g(t) \qquad \text{for all } t.$$

Let f be an arbitrary function. We have to show that

$$(f*g)(t+T) = (f*g)(t) \qquad \text{for all } t.$$

By definition

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

und therefore

$$(f*g)(t+T) = \int_{-\infty}^{\infty} f(\tau) g(t+T-\tau) d\tau.$$

As g is a T-periodic function, it holds that

$$g(t+T-\tau) = g(t-\tau)$$

und hence

$$\begin{aligned}
(f*g)(t+T) &= \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \\
&= (f*g)(t).
\end{aligned}$$

Exercise 4. Let $a \leq b$ and

$$g(t) = \begin{cases} 1 & \text{if } a \leq t \leq b \\ 0 & \text{else.} \end{cases}$$

Show that for any function f it holds that

$$(f*g)(t) = \int_{t-b}^{t-a} f(x) dx.$$

Solution for Exercise 4.

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \\ &= \int_{a}^{b} f(t-\tau)d\tau.\end{aligned}$$

Substitution

$$\begin{aligned}x &= t-\tau \\ \frac{dx}{d\tau} &= -1 \\ d\tau &= -dx\end{aligned}$$

It follows that

$$\begin{aligned}(f * g)(t) &= \int_{t-a}^{t-b} f(x)(-dx) \\ &= \int_{t-b}^{t-a} f(x)dx.\end{aligned}$$

Exercise 5. Show that convolution is commutative, i.e. for all functions f, g it holds that

$$f * g = g * f.$$

Solution for Exercise 5.

$$(f * g)(t) = \int_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

With substitution $\mu = t - \tau$ and $d\tau = -d\mu$ we obtain

$$\begin{aligned}\int_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)d\tau &= \int_{\mu=\infty}^{-\infty} -f(t-\mu)g(\mu)d\mu \\ &= \int_{\mu=-\infty}^{\infty} g(\mu)f(t-\mu)d\mu \\ &= \int_{\tau=-\infty}^{\infty} g(\tau)f(t-\tau)d\tau \\ &= (g * f)(t).\end{aligned}$$

Exercise 6. For $a > 0$ let index a denote the compression of a function by factor a, i.e.

$$f_a(t) = f(at).$$

Show that

$$f_a * g_a = \frac{1}{a}(f * g)_a.$$

Solution for Exercise 6.

$$\begin{aligned}(f_a * g_a)(t) &= \int_{-\infty}^{\infty} f_a(\tau)g_a(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(a\tau)g(a(t-\tau))d\tau \\ &= \int_{-\infty}^{\infty} f(a\tau)g(at-a\tau)d\tau.\end{aligned}$$

Substitution.

$$u = a\tau, \quad \frac{du}{d\tau} = a, \quad d\tau = \frac{1}{a}du.$$

We obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(a\tau)g(at-a\tau)d\tau &= \int_{-\infty}^{\infty} f(u)g(at-u)\frac{1}{a}du \\ &= \frac{1}{a}(f*g)(at) \\ &= \frac{1}{a}(f*g)_a(t).\end{aligned}$$

As this holds for all t, it follows that

$$f_a * g_a = \frac{1}{a}(f*g)_a.$$

Exercise 7. Let $f, g, h \in \mathbb{R} \to \mathbb{R}$ be functions. Function $g+h$ ist defined by

$$g+h \in \mathbb{R} \to \mathbb{R}, \quad (g+h)(t) = g(t) + h(t).$$

Show that

$$f*(g+h) = (f*g) + (f*h).$$

Solution for Exercise 7.

$$\begin{aligned}(f*(g+h))(t) &= \int_{-\infty}^{\infty} f(\tau)(g+h)(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)(g(t-\tau)+h(t-\tau))d\tau \\ &= \int_{-\infty}^{\infty} \big(f(\tau)g(t-\tau)+f(\tau)h(t-\tau)\big)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\ &= (f*g)(t) + (f*h)(t) \\ &= (f*g+f*h)(t).\end{aligned}$$

Exercise 8. Show that $\sigma * f$ is an antiderivative of f, i.e.

$$(\sigma * f)' = f.$$

This means that convolution with σ causes integration.

Solution for Exercise 8. Let F be an antiderivative of f. Then

$$\begin{aligned}(\sigma * f)(t) &= \int_{-\infty}^{\infty} \sigma(t-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{t} f(\tau) d\tau \\ &= F(t) - F(-\infty).\end{aligned}$$

Hence

$$\begin{aligned}(\sigma * f)'(t) &= (F(t) - F(-\infty))' \\ &= F'(t) \\ &= f(t).\end{aligned}$$

Exercise 9. Show that

$$(f * g)' = f * g'.$$

The proof is straight forward with Fourier Transform but it is a good exercise to do it in time domain as well.

Solution for Exercise 9. Proof in time domain:

$$\begin{aligned}(f*g)'(t) &= \frac{1}{dt}((f*g)(t+dt) - (f*g)(t)) \\ &= \frac{1}{dt}\left(\int_{-\infty}^{\infty} f(\tau)g(t+dt-\tau)d\tau - \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right) \\ &= \frac{1}{dt}\int_{-\infty}^{\infty} f(\tau)(g(t+dt-\tau) - g(t-\tau))d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\frac{g(t+dt-\tau) - g(t-\tau)}{dt}d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)g'(t-\tau)d\tau \\ &= (f*g')(t).\end{aligned}$$

Proof with Fourier Transform: From

$$f'(t) \quad \circ\!\!-\!\!\bullet \quad j\omega F(\omega)$$

and the Convolution Theorem it follows that

$$\begin{aligned}(f*g)'(t) \quad &\circ\!\!-\!\!\bullet \quad j\omega(F(\omega)G(\omega)) \\ &= \quad F(\omega)(j\omega G(\omega)) \\ &\bullet\!\!-\!\!\circ \quad (f*g')(t).\end{aligned}$$

Exercise 10. The Heaviside step function $\sigma(t)$ ist defined by

$$\sigma(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{else.} \end{cases}$$

- Let

$$\begin{aligned} f(t) &= \sigma(t)e^{at} \\ g(t) &= \sigma(t)e^{bt}. \end{aligned}$$

  Compute

$$(f * g)(t)$$

  and simplify the result as much as possible. Consider also the special case $a = b$.

- Use your result to compute $f * g$ for

$$\begin{aligned} f(t) &= \sigma(t)\sin(t) \text{ and} \\ g(t) &= \sigma(t)\cos(t). \end{aligned}$$

  Hint:

$$\sin(t) = \frac{1}{2j}\left(e^{jt} - e^{-jt}\right).$$

Solution for Exercise 10.

- Convolution of $f(t) = \sigma(t)e^{at}$ and $g(t) = \sigma(t)e^{bt}$.

$$\begin{aligned} &(f * g)(t) \\ &= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \sigma(\tau)e^{a\tau}\sigma(t-\tau)e^{b(t-\tau)}d\tau \\ &= \sigma(t)\int_{0}^{t} e^{a\tau}e^{b(t-\tau)}d\tau \\ &= \sigma(t)e^{bt}\int_{0}^{t} e^{a\tau}e^{-b\tau)}d\tau \\ &= \sigma(t)e^{bt}\int_{0}^{t} e^{(a-b)\tau}d\tau \\ &= \sigma(t)e^{bt}\frac{1}{a-b}[e^{(a-b)\tau}]_0^t, \qquad \text{if } a \neq b \\ &= \sigma(t)e^{bt}\frac{1}{a-b}(e^{(a-b)t}-1) \\ &= \sigma(t)\frac{1}{a-b}\left(e^{at}-e^{bt}\right) \end{aligned}$$

  If $a = b$ we obtain

$$\begin{aligned} (f * g)(t) &= \sigma(t)e^{at}\int_{0}^{t} e^{(a-a)\tau}d\tau \\ &= \sigma(t)e^{at}\int_{0}^{t} 1d\tau \\ &= \sigma(t)te^{at}. \end{aligned}$$

- Convolution of $f(t) = \sigma(t)\sin(t)$ and $g(t) = \sigma(t)\cos(t)$.

$$
\begin{aligned}
&(f * g)(t) \\
&= \sigma(t)\frac{1}{2j}\left(e^{jt} - e^{-jt}\right) * \sigma(t)\frac{1}{2}\left(e^{jt} + e^{-jt}\right) \\
&= \sigma(t)\frac{1}{4j}\left(e^{jt} * e^{jt} - e^{-jt} * e^{-jt} + \underbrace{e^{jt} * e^{-jt} - e^{-jt} * e^{jt}}_{=0}\right) \\
&= \sigma(t)\frac{1}{4j}\left(e^{jt} * e^{jt} - e^{-jt} * e^{-jt}\right) \\
&= \sigma(t)\frac{1}{4j}\left(te^{jt} - te^{-jt}\right) \\
&= \sigma(t)\frac{1}{2}t\sin(t).
\end{aligned}
$$