Heilbronn, 21.10.2025 Prof. Dr. V. Stahl WS 25/26

Homework for Digital Signal Processing with Solutions

Sheet 3

Exercise 1. Let

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

Compute the Fourier Transform $F(\omega)$ of $f(t)$.

Solution for Exercise 1.

$$\begin{aligned}
F(\omega) &= \int_0^1 f(t)e^{-j\omega t}dt \\
&= \int_0^1 te^{-j\omega t}dt \\
&= \left[-\frac{t}{j\omega}e^{-j\omega t}\right]_0^1 - \int_0^1 -\frac{1}{j\omega}e^{-j\omega t}dt \\
&= \frac{j}{\omega}\left[te^{-j\omega t}\right]_0^1 + \frac{1}{j\omega}\frac{-1}{j\omega}\left[e^{-j\omega t}\right]_0^1 \\
&= \frac{j}{\omega}e^{-j\omega} + \frac{1}{\omega^2}\left(e^{-j\omega} - 1\right) \\
&= \left(\frac{j}{\omega} + \frac{1}{\omega^2}\right)e^{-j\omega} - \frac{1}{\omega^2} \\
&= \frac{(j\omega + 1)e^{-j\omega} - 1}{\omega^2}
\end{aligned}$$

Exercise 2. Let

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)$$

and $a \in \mathbb{R}$ with $a > 0$. Show that

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{a}F\left(\frac{\omega}{a}\right).$$

Solution for Exercise 2.

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt.$$

With substitution

$$\tau = at, \quad \frac{d\tau}{dt} = a, \quad dt = \frac{1}{a}d\tau, \quad t = \frac{1}{a}\tau$$

it holds that

$$\begin{aligned}\int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt &= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/a}\,\frac{1}{a}d\tau \\ &= \frac{1}{a}\underbrace{\int_{-\infty}^{\infty} f(\tau)e^{-j(\omega/a)\tau}d\tau}_{F(\omega/a)} \\ &= \frac{1}{a}F\left(\frac{\omega}{a}\right).\end{aligned}$$

Exercise 3. Let $f \in \mathbb{R} \to \mathbb{R}$ and

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega).$$

Show that

$$f(-t) \quad \circ\!\!-\!\!\bullet \quad \overline{F(\omega)}.$$

Solution for Exercise 3.

$$f(-t) \quad \circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt.$$

With substitution

$$u = -t, \quad \frac{du}{dt} = -1, \quad dt = -du$$

we obtain

$$\begin{aligned}\int_{\infty}^{-\infty} f(u)e^{j\omega u}(-du) &= \int_{-\infty}^{\infty} f(u)e^{j\omega u}du \\ &= \int_{-\infty}^{\infty} f(u)\overline{e^{-j\omega u}}du \\ &= \overline{\int_{-\infty}^{\infty} f(u)e^{-j\omega u}du} \\ &= \overline{F(\omega)}.\end{aligned}$$

Exercise 4. Use the frequency shift correspondence

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \hat{\omega})$$

and

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

to determine the Fourier Transform of $e^{j\hat{\omega}t}$, $\cos(\hat{\omega}t)$ and $\sin(\hat{\omega}t)$.

Solution for Exercise 4. From the given correspondences it follows that

$$\begin{aligned}e^{j\hat{\omega}t} &\quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega - \hat{\omega}) \\ e^{-j\hat{\omega}t} &\quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega + \hat{\omega}).\end{aligned}$$

Using Euler's Theorem and linearity of the Fourier Transform we obtain

$$
\begin{aligned}
\cos(\hat{\omega}t) &= \frac{1}{2}(e^{j\hat{\omega}t} + e^{-j\hat{\omega}t}) \\
&\circ\!\!-\!\!\bullet \frac{1}{2}(2\pi\delta(\omega - \hat{\omega}) + 2\pi\delta(\omega + \hat{\omega})) \\
&= \pi(\delta(\omega - \hat{\omega}) + \delta(\omega + \hat{\omega})) \\
\sin(\hat{\omega}t) &= \frac{1}{2j}(e^{j\hat{\omega}t} - e^{-j\hat{\omega}t}) \\
&\circ\!\!-\!\!\bullet \frac{1}{2j}(2\pi\delta(\omega - \hat{\omega}) - 2\pi\delta(\omega + \hat{\omega})) \\
&= -j\pi(\delta(\omega - \hat{\omega}) - \delta(\omega + \hat{\omega}))
\end{aligned}
$$

Exercise 5. Use the Modulation Theorem and

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

to determine the Fourier Transform of $\cos(t)$.

Use the Time Shift Theorem and

$$\sin(t) \quad = \quad \cos(t - \pi/2)$$

to obtain the Fourier Transform of $\sin(t)$. Simplify the result as much as possible using the sifting property.

Use the correspondence

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}F(\omega/a)$$

and

$$\delta(at) \quad = \quad \frac{1}{|a|}\delta(t)$$

to derive the Fourier Transform of $\cos(\hat{\omega}t)$ from the Fourier Transform of $\cos(t)$ for arbitrary $\hat{\omega} \in \mathbb{R}$.

Solution for Exercise 5. From

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

and

$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega - \hat{\omega}) + F(\omega + \hat{\omega}))$$

it follows that

$$
\begin{aligned}
\cos(t) &\circ\!\!-\!\!\bullet \frac{1}{2}(2\pi\delta(\omega - 1) + 2\pi\delta(\omega + 1)) \\
&= \frac{1}{\pi}(\delta(\omega - 1) + \delta(\omega + 1)).
\end{aligned}
$$

From

$$f(t - \hat{t}) \quad \circ\!\!-\!\!\bullet \quad e^{-j\omega\hat{t}}F(\omega)$$

it follows with $\hat{t} = \pi/2$ that

$$\begin{aligned}\sin(t) &= \cos(t - \pi/2)\\ &\circ\!\!-\!\!\bullet\; e^{-j\omega\pi/2}\frac{1}{\pi}(\delta(\omega-1)+\delta(\omega+1))\\ &= \frac{1}{\pi}(e^{-j\omega\pi/2}\delta(\omega-1)+e^{-j\omega\pi/2}\delta(\omega+1))\\ &= \frac{1}{\pi}(e^{-j\pi/2}\delta(\omega-1)+e^{j\pi/2}\delta(\omega+1))\\ &= \frac{1}{\pi}(-j\delta(\omega-1)+j\delta(\omega+1))\\ &= -\frac{j}{\pi}(\delta(\omega-1)-\delta(\omega+1)).\end{aligned}$$

Finally, using

$$f(at) \;\circ\!\!-\!\!\bullet\; \frac{1}{|a|}F(\omega/a)$$

for $a = \hat{\omega}$ an $f(t) = \cos(t)$ we obtain

$$\begin{aligned}\cos(\hat{\omega}t) &\;\circ\!\!-\!\!\bullet\; \frac{1}{|\hat{\omega}|}\frac{1}{\pi}(\delta(\omega/\hat{\omega}-1)+\delta(\omega/\hat{\omega}+1))\\ &= \frac{1}{|\hat{\omega}|}\frac{1}{\pi}(|\hat{\omega}|\delta(\omega-\hat{\omega})+|\hat{\omega}|\delta(\omega+\hat{\omega}))\\ &= \frac{1}{\pi}(\delta(\omega-\hat{\omega})+\delta(\omega+\hat{\omega})).\end{aligned}$$

Exercise 6. Let $\omega = 2\pi/T$ and $u, v \in \mathbb{Z}$. Compute

$$\int_0^T e^{ju\omega t}e^{-jv\omega t}dt$$

and simplify the result as much as possible. Consider also the case $u = v$.

Solution for Exercise 6.

- Case $u \neq v$.

$$\begin{aligned}\int_0^T e^{ju\omega t}e^{-jv\omega t}dt &= \int_0^T e^{j(u-v)\omega t}dt\\ &= \frac{1}{j(u-v)\omega}[e^{j(u-v)\omega t}]_0^T\\ &= \frac{1}{j(u-v)\omega}\left(e^{j(u-v)\omega T}-1\right).\end{aligned}$$

  With $\omega T = 2\pi$ this is equal to

$$\frac{1}{j(u-v)\omega}\left(e^{2\pi j(u-v)}-1\right).$$

  As $u - v \in \mathbb{Z}$ it holds that

$$e^{2\pi j(u-v)} = 1.$$

  and therefore the integral is zero.

- Case $u = v$.

$$\begin{aligned}\int_0^T e^{ju\omega t} e^{-jv\omega t} dt &= \int_0^T e^{j(u-v)\omega t} dt \\ &= \int_0^T 1 dt \\ &= T.\end{aligned}$$

Summarizing we have

$$\int_0^T e^{ju\omega t} e^{-jv\omega t} dt = \begin{cases} 0 & \text{if } u \neq v \\ T & \text{if } u = v. \end{cases}$$

Exercise 7. Use the correspondences of the Fourier Transform to show that

$$f_{\hat{t}} * g = (f * g)_{\hat{t}}$$

where index $\hat{t}$ at a function means shifting the function by $\hat{t}$, i.e.

$$f_{\hat{t}}(t) = f(t - \hat{t}) \quad \text{for all } t.$$

The proof is very short.

Solution for Exercise 7. With

$$\begin{aligned} f(t) &\circ\!\!-\!\!\bullet\; F(\omega) \\ g(t) &\circ\!\!-\!\!\bullet\; G(\omega) \\ f_{\hat{t}}(t) &= f(t - \hat{t}) \\ &\circ\!\!-\!\!\bullet\; e^{-j\omega\hat{t}} F(\omega) \end{aligned}$$

it holds that

$$\begin{aligned} (f_{\hat{t}} * g)(t) &\circ\!\!-\!\!\bullet\; \left(e^{-j\omega\hat{t}} F(\omega)\right) G(\omega) \\ &= e^{-j\omega\hat{t}} F(\omega)(G\omega) \\ &\bullet\!\!-\!\!\circ\; (f * g)(t - \hat{t}) \\ &= (f * g)_{\hat{t}}(t). \end{aligned}$$

Exercise 8. Use the correspondences of the Fourier Transform to show that

$$f^- * g = (f * g^-)^-$$

where the raised minus sign means time reversal, i.e.

$$f^-(t) = f(-t).$$

The proof is very short.

Solution for Exercise 8. With

$$\begin{aligned} f^-(t) &= f(-t) \\ &\circ\!\!-\!\!\bullet\; \overline{F(\omega)} \\ (f * g)(t) &= F(\omega) G(\omega) \end{aligned}$$

and the law of complex numbers

$$\overline{z_1}\,\overline{z_2} = \overline{z_1 z_2}$$

it holds that

$$\begin{aligned}(f^- * g)(t) &\circ\!\!-\!\!\bullet \overline{F(\omega)}G(\omega)\\ &= \overline{F(\omega)}\,\overline{\overline{G(\omega)}}\\ &= \overline{F(\omega)\overline{G(\omega)}}\\ &\bullet\!\!-\!\!\circ (f * g^-)^-(t).\end{aligned}$$

Exercise 9. Let $f \in \mathbb{R} \to \mathbb{R}$ and

$$f^-(t) = f(-t)$$

for all t. The autocorrelation function f_A of f is defined by

$$f_A = f * f^-.$$

- Show that

$$f_A(0) = \int_{-\infty}^{\infty} f(t)^2 dt.$$

- Show that from $f^-(t) \circ\!\!-\!\!\bullet \overline{F(\omega)}$ and the convolution theorem it follows that

$$f_A(t) \circ\!\!-\!\!\bullet |F(\omega)|^2.$$

- The inverse Fourier Transform of $|F(\omega)|^2$ is

$$|F(\omega)|^2 \bullet\!\!-\!\!\circ \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega t} d\omega.$$

  From this we obtain

$$f_A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega t} d\omega.$$

  Show that in the special case $t = 0$ we obtain Parseval's equation

$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Solution for Exercise 9.

-

$$\begin{aligned}f_A(t) &= \int_{-\infty}^{\infty} f(\tau) f^-(t-\tau) d\tau\\ &= \int_{-\infty}^{\infty} f(\tau) f(\tau - t) d\tau.\end{aligned}$$

  F"ur $t = 0$ folgt

$$\begin{aligned}f_A(0) &= \int_{-\infty}^{\infty} f(\tau) f(\tau) d\tau\\ &= \int_{-\infty}^{\infty} f(t)^2 dt.\end{aligned}$$

- $$\begin{aligned} f_A(t) &= (f * f^-)(t) \\ &\circ\!\!-\!\!\bullet\ F(\omega)\overline{F(\omega)} \\ &= |F(\omega)|^2. \end{aligned}$$

- Aus

$$f_A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega t} d\omega$$

folgt f"ur $t = 0$

$$\begin{aligned} f_A(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega 0} d\omega \\ \int_{-\infty}^{\infty} f(t)^2 dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega. \end{aligned}$$

Exercise 10. Compute the Fourier Transform of

$$\cos(2t)\sin(3t)$$

in three different ways:

- using complex exponential functions for the cosine- and sine function.
- using the modulation theorem

$$f(t)\cos(\hat{\omega})t \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}\left(F(\omega-\hat{\omega}) + F(\omega+\hat{\omega})\right).$$

- using the convolution theorem in frequency domain

$$f(t)g(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * G)(\omega).$$

Solution for Exercise 10.

Using complex exponentials.

$$\begin{aligned} \cos(2t)\sin(3t) &= \frac{1}{2}\left(e^{2jt} + e^{-2jt}\right) + \frac{1}{2j}\left(e^{3jt} - e^{-3jt}\right) \\ &= \frac{1}{4j}\left(e^{5jt} - e^{-jt} + e^{jt} - e^{-5jt}\right) \\ &\circ\!\!-\!\!\bullet\ \frac{2\pi}{4j}\Big(\delta(\omega-5) - \delta(\omega+1) + \delta(\omega-1) - \delta(\omega+5)\Big) \\ &= \frac{j\pi}{2}\Big(\delta(\omega+5) - \delta(\omega-5) + \delta(\omega+1) - \delta(\omega-1)\Big) \end{aligned}$$

Using modulation theorem. From the table of correspondences in the lecture notes we obtain

$$\sin(3t) \quad \circ\!\!-\!\!\bullet \quad -j\pi(\delta(\omega-3) - \delta(\omega+3)).$$

From the modulation theorem it follows that

$$\begin{aligned} \cos(2t)\sin(3t) &\circ\!\!-\!\!\bullet\ -\frac{j\pi}{2}\left(\delta(\omega-2-3) - \delta(\omega-2+3) + \delta(\omega+2-3) - \delta(\omega+2+3)\right) \\ &= \frac{j\pi}{2}\left(\delta(\omega+5) - \delta(\omega-5) + \delta(\omega+1) - \delta(\omega-1)\right) \end{aligned}$$

Using convolution theorem in frequency domain. Sei

$$\begin{aligned} f(t) &= \cos(2t) \\ g(t) &= \sin(3t). \end{aligned}$$

From the table of correspondences in the lecture notes we obtain

$$\begin{aligned} \cos(2t) &\circ\!\!-\!\!\bullet\ \pi(\delta(\omega-2)+\delta(\omega+2)) \\ \sin(3t) &\circ\!\!-\!\!\bullet\ -j\pi(\delta(\omega-3)-\delta(\omega+3)). \end{aligned}$$

It follows that

$$\begin{aligned} & f(t)g(t) \\ \circ\!\!-\!\!\bullet\ & \frac{1}{2\pi}(F*G)(\omega) \\ =\ & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)G(\omega-u)du \\ =\ & \frac{1}{2\pi}(-j\pi^2)\int_{-\infty}^{\infty}\Big(\delta(u-2)+\delta(u+2)\Big)\Big(\delta(\omega-u-3)-\delta(\omega-u+3)\Big)du \\ =\ & -\frac{j\pi}{2}\int_{-\infty}^{\infty}\Big(\delta(u-2)\delta(\omega-u-3)-\delta(u-2)\delta(\omega-u+3) \\ & +\delta(u+2)\delta(\omega-u-3)-\delta(u+2)\delta(\omega-u+3)\Big)du \end{aligned}$$

The integrals are solved separately as follows:

$$\begin{aligned} \int_{-\infty}^{\infty}\delta(u-2)\delta(\omega-u-3)du &= \int_{-\infty}^{\infty}\delta(u-2)\delta(\omega-5)du \\ &= \delta(\omega-5) \\ \int_{-\infty}^{\infty}\delta(u-2)\delta(\omega-u+3)du &= \int_{-\infty}^{\infty}\delta(u-2)\delta(\omega+1)du \\ &= \delta(\omega+1) \\ \int_{-\infty}^{\infty}\delta(u+2)\delta(\omega-u-3)du &= \int_{-\infty}^{\infty}\delta(u+2)\delta(\omega-1)du \\ &= \delta(\omega-1) \\ \int_{-\infty}^{\infty}\delta(u+2)\delta(\omega-u+3)du &= \int_{-\infty}^{\infty}\delta(u+2)\delta(\omega+5) \\ &= \delta(\omega+5) \end{aligned}$$

Hence

$$\begin{aligned} f(t)g(t)\ \circ\!\!-\!\!\bullet\ & -\frac{j}{2\pi}\Big(\delta(\omega-5)-\delta(\omega+1)+\delta(\omega-1)-\delta(\omega+5)\Big) \\ =\ & \frac{j}{2\pi}\Big(\delta(\omega+5)-\delta(\omega-5)+\delta(\omega+1)-\delta(\omega-1)\Big). \end{aligned}$$

Exercise 11. Let

$$f(t) = e^{j\omega t}$$

be a complex oscillation with constant angular frequency ω. Show that

$$(f * g)(t) = G(\omega)e^{j\omega t}$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau}d\tau$$

is the Fourier Transform of $g(t)$.

This means that a linear, time invariant systems answeres to an oscillation with angular frequency ω always with an oscillation with the same angular frequency ω. The oscillation is merely multiplied by a constant (time independent) factor $G(\omega)$. As $G(\omega)$ is complex this means a phase shift and an amplification of the oscillation.

Solution for Exercise 11.

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}g(\tau)d\tau \\ &= e^{j\omega t}\int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau}d\tau \\ &= G(\omega)e^{j\omega t}.\end{aligned}$$

Exercise 12. Figure 1 shows the Fourier Transform $F(\omega)$ of a band limited signal $f(t)$ with cut-off frequency $\hat{\omega}$.

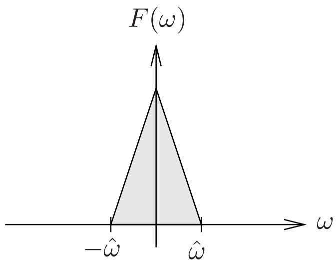

Figure 1: Fourier Transform of $f(t)$.

Let

$$f(t)\cos(\omega_0 t) \quad \circ\!\!-\!\!\bullet \quad F_{\text{mod}}(\omega).$$

where $\omega_0 > \hat{\omega}$.

- Make a sketch of $F_{\text{mod}}(\omega)$. Use the frequency shifting property of Fourier Transform.

- What might be the purpose to multiply a signal $f(t)$ before its transmission by $\cos(\omega_0 t)$?
- What is the result if the signal is multplied a second time by $\cos(\omega_0 t)$ and then convolved with a function $g(t)$ whose Fourier Transform is

$$G(\omega) = \begin{cases} 2 & \text{if } -\hat{\omega} \leq \omega \leq \hat{\omega} \\ 0 & \text{else?} \end{cases}$$

Solution for Exercise 12. According to the frequency shifting property of Fourier Transform it holds that

$$f(t)e^{j\omega_0 t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \omega_0).$$

From

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

it follows that

$$\begin{aligned} f(t)\cos(\omega_0 t) &= \frac{1}{2}f(t)e^{j\omega_0 t} + \frac{1}{2}f(t)e^{-j\omega_0 t} \\ &\circ\!\!-\!\!\bullet \quad \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0) \\ &= F_{\text{mod}}(\omega). \end{aligned}$$

This means that $F_{\text{mod}}(\omega)$ consists of two copies of $F(\omega)$ with half amplitude each and shifted by ω_0 to the left and to the right, see Figure 2. As $\omega_0 > \hat{\omega}$, the copies do not overlap.

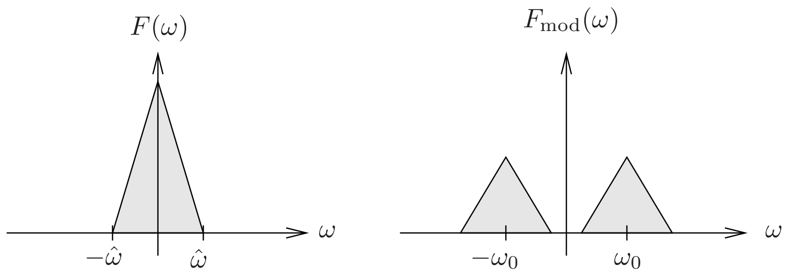

Figure 2: Left: Fourier Transform of $f(t)$. Right: Fourier Transform of $f(t)\cos(\omega_0 t)$.

A second multiplication by $\cos(\omega_0 t)$ gives

$$\begin{aligned} & f(t)\cos(\omega_0 t)\cos(\omega_0 t) \\ \circ\!\!-\!\!\bullet \quad & \frac{1}{2}F_{\text{mod}}(\omega - \omega_0) + \frac{1}{2}F_{\text{mod}}(\omega + \omega_0) \\ = \quad & \frac{1}{4}F(\omega - 2\omega_0) + \frac{1}{4}F(\omega) + \frac{1}{4}F(\omega) + \frac{1}{4}F(\omega + 2\omega_0) \\ = \quad & \frac{1}{4}F(\omega - 2\omega_0) + \frac{1}{2}F(\omega) + \frac{1}{4}F(\omega + 2\omega_0) \\ = \quad & F_{\text{rek}}(\omega), \end{aligned}$$

see Figure 3.

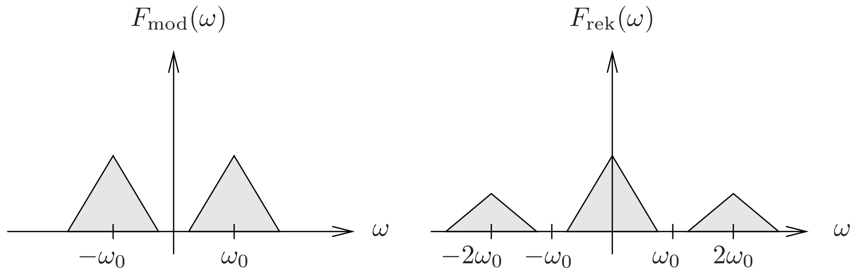

Figure 3: Left: Fourier Transform of $f(t)\cos(\omega_0 t)$. Right: Fourier Transform of $f(t)\cos(\omega_0 t)\cos(\omega_0 t)$.

Low pass filtering with $G(\omega)$ reconstructs in frequency domain $F(\omega)$ and correspondingly in time domain the original $f(t)$. This method is called amplitude modulation and is used when several signals are transmitted over the same medium. Each signal is assigned its exclusive frequency band (here $[\omega_0 - \hat{\omega}, \omega_0 + \hat{\omega}]$).