Heilbronn, 28.10.2025 Prof. Dr. V. Stahl WS 25/26

Homework for Digital Signal Processing with Solutions

Sheet 4

Exercise 1. Let $F(\omega)$ be the Fourier Transform of $f(t)$. What do the following operations and assumptions on $f(t)$ mean for $F(\omega)$?

- $f(t)$ is multiplied by a T_s-periodic pulse train.
- $f(t)$ is T_0-periodic.
- $f(t)$ is band limited with cut-off frequency $\hat{\omega}$.

Solution for Exercise 1.

- Multiplication of $f(t)$ with a T_s periodic pulse train corresponds to the ω_s-periodic continuation of $F(\omega)$ and scaling by $1/T_s$.
- If $f(t)$ is periodic with period T_0 then $F(\omega)$ consists of Dirac pulses with distance ω_0. The amplitudes of the pulse at $\omega = k\omega_0$ is proportional to the j-th Fourier coefficient z_k of f.
- If $f(t)$ is band limited with cut-off frequency $\hat{\omega}$, then $F(\omega) = 0$ for $|\omega| > \hat{\omega}$.

Exercise 2. Show that discrete convolution is commutative, i.e.

$$\sum_{k=-\infty}^{\infty} f_k g_{\ell-k} = \sum_{k=-\infty}^{\infty} g_k f_{\ell-k}.$$

Solution for Exercise 2. With substitution $k' = \ell - k$ (or $k = \ell - k'$) we obtain

$$\sum_{k=-\infty}^{\infty} f_k g_{\ell-k} = \sum_{k'=\infty}^{-\infty} f_{\ell-k'} g_{k'}.$$

Substitution $k = k'$ gives

$$\sum_{k=\infty}^{-\infty} f_{\ell-k} g_k = \sum_{k=-\infty}^{\infty} f_{\ell-k} g_k = \sum_{k=-\infty}^{\infty} g_k f_{\ell-k}.$$

Note that in contrast to integrals there is no dt in sums and thanks to commutativity the order of the summands is arbitrary.

Exercise 3. (Quadrature Amplitude Modulation.) The modulation theorem of Fourier Transform says that

$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega - \hat{\omega}) + F(\omega + \hat{\omega})).$$

If you consider the derivation of this theorem you can show in the same way that

$$f(t)\sin(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2j}(F(\omega-\hat{\omega}) - F(\omega+\hat{\omega})).$$

These theorems can be used to transmit *two* real signals $f(t)$ and $g(t)$ simultaneously in the same frequency band. This method is called quadrature amplitude modulation.

- The sender modulates $f(t)$ with $\cos(\hat{\omega}t)$ and $g(t)$ with $\sin(\hat{\omega}t)$ and adds both signals.
- If the receiver wants to obtain $f(t)$, he demodulates with $\cos(\hat{\omega}t)$.
- If the receiver wants to obtain $g(t)$, he demodulates with $\sin(\hat{\omega}t)$.

This can be seen as follows: The sender signal with the modulated $f(t)$ and $g(t)$ is

$$h(t) \quad = \quad f(t)\cos(\hat{\omega}t) + g(t)\sin(\hat{\omega}t).$$

- Show that

  $$h(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet$$
  $$\frac{1}{2}F(\omega) + \frac{1}{4}\underbrace{(F(\omega-2\hat{\omega}) + F(\omega+2\hat{\omega}) + jG(\omega-2\hat{\omega}) - jG(\omega+2\hat{\omega}))}_{\text{components in the } 2\hat{\omega} \text{ band}}$$

  After lowpass filtering and multiplication with 2 we obtain $f(t)$.

- Show that

  $$h(t)\sin(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet$$
  $$\frac{1}{2}G(\omega) + \frac{1}{4}\underbrace{(-jF(\omega-2\hat{\omega}) + jF(\omega+2\hat{\omega}) - G(\omega-2\hat{\omega}) - G(\omega+2\hat{\omega}))}_{\text{components in the } 2\hat{\omega} \text{ band}}$$

  After lowpass filtering and multiplication with 2 we obtain $g(t)$.

Solution for Exercise 3. It holds that

$$\begin{aligned} f(t)e^{j\hat{\omega}t} \quad &\circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} f(t)e^{j\hat{\omega}t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\hat{\omega})t}dt \\ &= F(\omega-\hat{\omega}). \end{aligned}$$

Hence

$$\begin{aligned}
f(t)\cos(\hat\omega t) &= f(t)\frac{e^{j\hat\omega t}+e^{-j\hat\omega t}}{2} \\
&= \frac{1}{2}\big(f(t)e^{j\hat\omega t}+f(t)e^{-j\hat\omega t}\big) \\
&\circ\!\!-\!\!\bullet\ \frac{1}{2}(F(\omega-\hat\omega)+F(\omega+\hat\omega)) \\
f(t)\sin(\hat\omega t) &= f(t)\frac{e^{j\hat\omega t}-e^{-j\hat\omega t}}{2j} \\
&= \frac{1}{2j}\big(f(t)e^{j\hat\omega t}-f(t)e^{-j\hat\omega t}\big) \\
&\circ\!\!-\!\!\bullet\ \frac{1}{2j}(F(\omega-\hat\omega)-F(\omega+\hat\omega))
\end{aligned}$$

Therefore

$$\begin{aligned}
h(t) &= f(t)\cos(\hat\omega t)+g(t)\sin(\hat\omega t) \\
&\circ\!\!-\!\!\bullet\ \frac{1}{2}(F(\omega-\hat\omega)+F(\omega+\hat\omega))+\frac{1}{2j}(G(\omega-\hat\omega)-G(\omega+\hat\omega)) \\
&= \frac{1}{2}(F(\omega-\hat\omega)+F(\omega+\hat\omega))-jG(\omega-\hat\omega)+jG(\omega+\hat\omega)) \\
&= H(\omega).
\end{aligned}$$

Further we obtain

$$\begin{aligned}
&h(t)\cos(\hat\omega t)\ \circ\!\!-\!\!\bullet\ \frac{1}{2}(H(\omega-\hat\omega)+H(\omega+\hat\omega)) \\
&= \frac{1}{4}(F(\omega-2\hat\omega)+F(\omega)-jG(\omega-2\hat\omega)+jG(\omega)\ + \\
&\quad F(\omega)+F(\omega+2\hat\omega)-jG(\omega)+jG(\omega+2\hat\omega)) \\
&= \frac{1}{2}F(\omega)+\frac{1}{4}(F(\omega-2\hat\omega)+F(\omega+2\hat\omega)-jG(\omega-2\hat\omega)+jG(\omega+2\hat\omega))
\end{aligned}$$

and

$$\begin{aligned}
&h(t)\sin(\hat\omega t)\ \circ\!\!-\!\!\bullet\ \frac{1}{2j}(H(\omega-\hat\omega)-H(\omega+\hat\omega)) \\
&= \frac{1}{4j}(F(\omega-2\hat\omega)+F(\omega)-jG(\omega-2\hat\omega)+jG(\omega)\ - \\
&\quad (F(\omega)+F(\omega+2\hat\omega)-jG(\omega)+jG(\omega+2\hat\omega))) \\
&= \frac{1}{4j}(F(\omega-2\hat\omega)+F(\omega)-jG(\omega-2\hat\omega)+jG(\omega)\ - \\
&\quad F(\omega)-F(\omega+2\hat\omega)+jG(\omega)-jG(\omega+2\hat\omega)) \\
&= \frac{1}{2}G(\omega)+\frac{1}{4j}(F(\omega-2\hat\omega)-F(\omega+2\hat\omega)-jG(\omega-2\hat\omega)-jG(\omega+2\hat\omega)) \\
&= \frac{1}{2}G(\omega)+\frac{1}{4}(-jF(\omega-2\hat\omega)+jF(\omega+2\hat\omega)-G(\omega-2\hat\omega)-G(\omega+2\hat\omega)).
\end{aligned}$$

Exercise 4. In the lecture we derived the Fourier series of the pulse train $p(t)$:

$$\begin{aligned} p(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}, \quad \omega_s = 2\pi/T_s. \end{aligned}$$

Use this representation to compute the Fourier Transform $P(\omega)$ of $p(t)$ and show that $P(\omega)$ is also a pulse train. You may use the correspondence

$$e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega - \hat{\omega}).$$

Solution for Exercise 4. From linearity of Fourier Transform it follows that

$$\begin{aligned} p(t) &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \\ &\circ\!\!-\!\!\bullet \ \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_s) \\ &= \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s). \end{aligned}$$

This is a pulse train where the distance between the pulses is ω_s and their magnitude is ω_s.

Exercise 5. In the lecture we computed the Fourier Transform $F_p(\omega)$ of $f_p(t) = f(t)p(t)$ using the frequency shift property of the Fourier Transform:

$$f_p(t) = f(t)p(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s) = F_p(\omega).$$

An alternative way would have been to derive $F_p(\omega)$ using the convolution theorem in frequency domain:

$$f(t)p(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * P)(\omega).$$

In a previous exercise we computed $P(\omega)$. Now compute the convolution $(F * P)(\omega)$ and show that in fact

$$F_p(\omega) = \frac{1}{2\pi}(F * P)(\omega).$$

Solution for Exercise 5. From the previous exercise we obtain

$$P(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

It follows that

$$\begin{aligned}
\frac{1}{2\pi}(F*P)(\omega) &= \int_{-\infty}^{\infty} F(u)P(\omega-u)du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\,\underbrace{\omega_s \sum_{k=-\infty}^{\infty} \delta(\omega-u-k\omega_s)}_{P(\omega-u)}\,du \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} F(u)\delta(\omega-k\omega_s-u)du \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega-k\omega_s)\delta(\omega-k\omega_s-u)du \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega-k\omega_s)\int_{-\infty}^{\infty} \delta(\omega-k\omega_s-u)du \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega-k\omega_s).
\end{aligned}$$

The same result can be obtained in a more straight forward way by using linearity of convolution and the fact that convolution with a shifted pulse causes a shift in the result:

$$\begin{aligned}
\frac{1}{2\pi}(F*P)(\omega) &= \frac{1}{2\pi}F(\omega)*\omega_s\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s) \\
&= \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} F(\omega)*\delta(\omega-k\omega_s) \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega-k\omega_s)
\end{aligned}$$

Exercise 6. (Sampling rate conversion) The formula

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \mathrm{sinc}(n-t/T_s)$$

was derived in the lecture and used to reconstruct the analog signal $f(t)$ from samples f_n.

It can also be applied to change the sampling rate. Derive a formula which takes samples f_n with sampling rate ω_s and returns new samples f'_n of the same signal $f(t)$ but with a different sampling rate ω'_s.

You may assume that $f(t)$ is band limited with cut-off frequency $\hat{\omega}$ and the the old and new sampling rate ω_s and ω'_s are larger than $2\hat{\omega}$.

Choose a suitable approximation to truncate the infinite sum to a finite sum with $2N+1$ summands and implement a function to change the sampling rate. The function takes a finite sequence f_n of samples as well as the old and new sampling rate ω_s and ω_s' and returns a new sequence of samples f_n'.

Verify your program with a simple function like $f(t) = \cos(\hat{\omega}t)$ and $\omega_s, \omega_s' > 2\hat{\omega}$. The samples are

$$\begin{aligned} f_\ell &= f(\ell T_s) \\ &= \cos(\hat{\omega}\ell T_s). \end{aligned}$$

The samples with new sampling rate ω_s' are

$$\begin{aligned} f_\ell' &\approx f(\ell T_s') \\ &= \cos(\hat{\omega}\ell T_s'). \end{aligned}$$

Verify that the approximation is more accurate as N increases.

In order to avoid boundary effects, it is assumed that f_ℓ and f_ℓ' are infinitely long, but of course only finitely many samples of f_ℓ' are actually computed.

Solution for Exercise 6.

$$\begin{aligned} f_\ell' &= f(\ell T_s') \\ &= \sum_{n=-\infty}^{\infty} f_n \text{sinc}(n - \ell T_s'/T_s) \\ &= \sum_{n=-\infty}^{\infty} f_n \text{sinc}(n - \ell \omega_s/\omega_s') \\ &= \sum_{n=-\infty}^{\infty} f_n \text{sinc}(n - \ell\beta) \end{aligned}$$

where

$$\beta = \frac{\omega_s}{\omega_s'}.$$

The sinc-function gives significant values if

$$\ell\beta \approx n.$$

Let

$$n_0 = \text{round}\left(\ell\beta\right).$$

Then

$$f_\ell' \approx \sum_{n=n_0-N}^{n_0+N} f_n \text{sinc}(n - \ell\beta).$$

The approximation is more accurate for larger values of N. On the other hand, a large value of N costs more computing time.

Exercise 7. The discrete convolution of two sequences f, g is definied by

$$(f * g)_\ell \quad = \quad \sum_{k=-\infty}^{\infty} g_k f_{\ell-k}.$$

If f and g are causal, this simplifies to

$$(f * g)_\ell \quad = \quad \sum_{k=0}^{\ell} g_k f_{\ell-k}.$$

If g has finite length N (FIR filter), the formula simplifies further to

$$(f * g)_\ell \quad = \quad \sum_{k=0}^{N-1} g_k f_{\ell-k}.$$

This means that for each sample ℓ the same number N of multiplications and $N-1$ of additions have to be performed and the filter can operate in real time.

Implement a function which takes two causal sequences f, g and an integer ℓ as input and returns $(f * g)_\ell$. Test your program e.g. by choosing one signal as the Dirac pulse or a shifted Dirac pulse. The discrete Dirac pulse is defined as

$$\delta_k \quad = \quad \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0. \end{cases}$$

Solution for Exercise 7. Programming exercise.

Exercise 8. Generate a signal which consists of two cosine functions with different frequencies ω_1 and ω_2 where $\omega_1 < \omega_2$:

$$\begin{aligned} f_1(t) &= \cos(\omega_1 t) \\ f_2(t) &= \cos(\omega_2 t) \\ f(t) &= f_1(t) + f_2(t). \end{aligned}$$

Generate samples of f with sampling interval T_s:

$$f_k \quad = \quad f(kT_s).$$

Make sure that the samping rate $\omega_s = {}^{2\pi}/_{T_s}$ is sufficiently high, i.e.

$$\omega_s \quad > \quad 2\omega_2.$$

Compute samples h_ℓ of the lowpass filtered signal $h(t)$ with cutoff frequency ω_c by discrete convolution

$$h_\ell \quad = \quad \sum_{k=-N}^{N} g_k f_{\ell-k}$$

where

$$g_k \quad = \quad \hat{\omega}_c \mathrm{sinc}(k\hat{\omega}_c), \qquad \hat{\omega}_c \quad = \quad \frac{2\omega_c}{\omega_s}, \qquad k = \; -N \ldots, N$$

are the coefficients of the truncated lowpass filter.

Choose the cutoff frequency ω_c such that

$$\omega_1 < \omega_c < \omega_2.$$

In this case the high frequency signal f_2 is filtered out and f_1 remains unchanged. Large values for N (like $N > 50$) lead to more accurate results, but take more computing time. Verify your results by comparing plots of samples of f and h. Try different values for N, ω_1, ω_2, ω_c and ω_s.

In order to avoid boundary problems you may assume that f has infinite lenght, although only finitely many samples $f_{\ell-N}, \ldots, f_{\ell+N}$ are actually needed to compute h_ℓ.

Solution for Exercise 8. Programming exercise.