Homework for Digital Signal Processing with Solutions

Sheet 6

Exercise 1. Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e. a matrix whose columns are pairwise orthogonal unit vectors which means

$$\vec{a}_i \circ \vec{a}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{else} \end{cases}$$

where $\vec{a}_i$ and $\vec{a}_j$ are the i-th and j-th column vector of A. Show that it holds that

$$A^{\mathrm{T}} A = E$$

where E is the $n \times n$ unit matix.

Solution for Exercise 1. For the multiplication of two matrices $A, B \in \mathbb{R}^{n\times n}$ it holds that

$$(AB)_{ij} = \ i\text{-th row of } A \text{ times } j\text{-th column of } B.$$

The i-th row of A^{T} has the same components as the i-th column of A. Therefore the i-th row of A is equal to $\vec{a}_i^{\mathrm{T}}$ and

$$\begin{aligned} (A^{\mathrm{T}}A)_{ij} &= \vec{a}_i^{\mathrm{T}} \vec{a}_j \\ &= \vec{a}_i \circ \vec{a}_j \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{else} \end{cases} \end{aligned}$$

This means that all elements of $(A^{\mathrm{T}}A)$ are zero except for the diagonal which is one, hence

$$A^{\mathrm{T}}A = E.$$

This result is obtained in a formal way as follows:

$$\begin{aligned} (A^{\mathrm{T}}A)_{ij} &= \sum_{k=1}^{n} (A^{\mathrm{T}})_{ik} A_{kj} \\ &= \sum_{k=1}^{n} A_{ki} A_{kj} \\ &= \vec{a}_i \circ \vec{a}_j \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{else} \end{cases} \end{aligned}$$

Exercise 2. Let $\vec{x}, \vec{y} \in \mathbb{C}^n$ be vectors with components

$$\begin{aligned} x_k &= e^{2\pi j u k/n} \\ y_k &= e^{2\pi j v k/n} \end{aligned}$$

for $k = 0, 1, \ldots, n-1$. The complex inner product of vectors is defined as

$$\vec{x} \circ \vec{y} = \sum_{k=0}^{n-1} \overline{x_k} y_k.$$

Show that for arbitrary $u, v \in \mathbb{N}_0$ it holds that

$$\vec{x} \circ \vec{y} = \begin{cases} 0 & \text{if } u \neq v \\ n & \text{if } u = v. \end{cases}$$

Hint: Try to rewrite the inner product as

$$\vec{x} \circ \vec{y} = \sum_{k=0}^{n-1} a^k$$

for a suitable a and use the formula

$$\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1}.$$

Solution for Exercise 2.

$$\begin{aligned} \vec{x} \circ \vec{y} &= \sum_{k=0}^{n-1} \overline{x_k} y_k \\ &= \sum_{k=0}^{n-1} e^{-2\pi j u k/n} e^{2\pi j v k/n} \\ &= \sum_{k=0}^{n-1} e^{-2\pi j (v-u) k/n} \\ &= \sum_{k=0}^{n-1} \big(\underbrace{e^{-2\pi j (v-u)/n}}_{a}\big)^k \\ &= \sum_{k=0}^{n-1} a^k \end{aligned}$$

for

$$a = e^{-2\pi j (v-u)/n}.$$

- If $u = v$ then $a = 1$ and therefore $\vec{x} \circ \vec{y} = n$.
- If $u \neq v$ let $s = \vec{x} \circ \vec{y}$ and hence

$$s = \sum_{k=0}^{n-1} a^k.$$

Multipliying both sides by a we obtain

$$\begin{aligned} as &= a\sum_{k=0}^{n-1} a^k \\ &= \sum_{k=0}^{n-1} a^{k+1} \\ &= \sum_{k=1}^{n} a^k. \end{aligned}$$

From this we obtain

$$\begin{aligned} as - s &= \sum_{k=1}^{n} a^k - \sum_{k=0}^{n-1} a^k \\ &= a^n - 1 \end{aligned}$$

and therefore

$$\begin{aligned} s(a-1) &= a^n - 1 \\ s &= \frac{a^n - 1}{a-1}. \end{aligned}$$

With the above choice of a we obtain

$$\begin{aligned} a^n &= \left(e^{-2\pi j(v-u)/n}\right)^n \\ &= e^{-2\pi j(v-u)} \\ &= 1. \end{aligned}$$

Hence

$$s = \frac{a^n - 1}{a-1} = 0.$$

Exercise 3. Write a program which computes matrix $B \in \mathbb{C}^{n\times n}$ for an arbitray $n \in \mathbb{N}$ where

$$b_{k\ell} = e^{2\pi jk\ell/n}.$$

Solution for Exercise 3. Programming exercise.

Exercise 4. Write a program which computes for given Fourier coefficients $\vec{z} \in \mathbb{C}^n$ the corresponding samples $\vec{f}$ by matrix vector multiplication

$$\vec{f} = B\vec{z}.$$

- The Fourier coefficients $\vec{z}$ appear in conjugate complex pairs. Make some tests that $\vec{f}$ is real if $\vec{z}$ satisfies

$$z_{n-k} = \overline{z_k} \text{ for all } k = 1, \dots, n-1.$$

- Test that $f_k = A_0$ for all $k = 0, \dots, n-1$, i.e. the signal is constant if

$$z_0 = A_0, \quad z_k = 0 \text{ for } k = 1, \dots, n-1.$$

- The Fourier coefficients are given by

$$z_k = \frac{1}{2} A_k e^{j\varphi_k}, \quad k = 1, 2, \ldots, n/2 - 1$$

where A_k is the amplitude of the k-th harmonic of $f(t)$ and φ_k is its phase. Verify that the samples $\vec{f}$ are actually m periods of a cosine wave with amplitude 2 if

$$z_k = \begin{cases} 1 & \text{if } k = m \text{ or } k = n - m \\ 0 & \text{else} \end{cases}$$

for $k = 1, \ldots, n/2 - 1$.

Solution for Exercise 4. Programming exercise.

Exercise 5. Implement a function for DFT and IDFT. Test with some random vectors $\vec{z} \in \mathbb{C}^n$ and $\vec{f} \in \mathbb{R}^n$ that

$$\begin{aligned} \vec{f} &= \text{IDFT}(\text{DFT}(\vec{f})) \\ \vec{z} &= \text{DFT}(\text{IDFT}(\vec{z})). \end{aligned}$$

Due to rounding errors of floating point arithmetic small deviations may occur. Do not forget the factor $1/n$!

Solution for Exercise 5. Programming exercise.

Exercise 6. Let

$$f(t) = 3 + \cos(t + 1) + 2\cos(3t + 2) - 5\cos(4t - 1)$$

be a $T_0 = 2\pi$ periodic function. Sample $f(t)$ at 16 equidistant points in the interval $[0, 2\pi]$ and form a vector $\vec{f} \in \mathbb{R}^{16}$. What are the corresponding Fourier coefficients $\vec{z} \in \mathbb{C}^{16}$? Use the program from the previous exercise to verify that the DFT actually gives those values.

Solution for Exercise 6. For the Fourier coefficients z_k it holds in general that

$$\begin{aligned} z_0 &= A_0 \\ z_k &= \frac{1}{2} A_k e^{j\varphi_k}, \quad k = 1, \ldots, n/2 - 1 \\ z_{n/2} &= 0 \\ z_k &= \overline{z_{n-k}}, \quad k = n/2 + 1, \ldots, n - 1. \end{aligned}$$

For the given example it follows that

$$\begin{aligned} z_0 &= 3 \\ z_1 &= \frac{1}{2} e^{j} \\ z_3 &= e^{2j} \\ z_4 &= \frac{5}{2} e^{j(\pi - 1)} \\ z_{12} &= \frac{5}{2} e^{-j(\pi - 1)} \\ z_{13} &= e^{-2j} \\ z_{15} &= \frac{1}{2} e^{-j} \end{aligned}$$

and all other $z_k = 0$.

Exercise 7. The matrices B and B^* consume a lot of memory for large n. However, the entries of the ℓ-th row of B can be obtained from the entries of row $\ell = 1$ of B by selecting every ℓ-the element. Prove that

$$B_{\ell k} \quad = \quad B_{1,(k\ell) \bmod n}$$

for all $\ell, k = 0, 1, \ldots, n-1$.

Here mod is the modulo operation and

$$(k\ell) \bmod n \quad = \quad k\ell - un$$

where $u \in \mathbb{Z}$ is such that

$$0 \leq k\ell - un < n.$$

The same property holds for B^*. Implement the DFT and IDFT in a memory efficient way where only row $\ell = 1$ of B is stored.

Solution for Exercise 7. Let $u \in \mathbb{Z}$ such that

$$0 \leq k\ell - un < n.$$

Then

$$\begin{aligned}
B_{1,(k\ell) \bmod n} &= e^{2\pi j (k\ell) \bmod n / n} \\
&= e^{2\pi j (k\ell - un)/n} \\
&= e^{2\pi j k\ell/n} e^{-2\pi j un/n} \\
&= e^{2\pi j k\ell/n} \underbrace{e^{-2\pi j u}}_{=1} \\
&= e^{2\pi j k\ell/n} \\
&= B_{\ell k}.
\end{aligned}$$

Exercise 8. Let

$$p(t) \quad = \quad \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T_s)$$

be a pulse train with period T_s.

Assume $f(t)$ is T_0-periodic with $T_0 = nT_s$.

Show that $f(t)p(t)$ is also T_0 periodic.

Compute the Fourier coefficients z_k of $f(t)p(t)$ by integration and show that

$$z_k T_s \quad = \quad \underbrace{\frac{1}{n} \sum_{\ell=0}^{n-1} f_\ell e^{-2\pi j k\ell/n}}_{\text{DFT of } \vec{f}}$$

where $f_\ell = f(\ell T_s)$. This means that the DFT of the samples f_ℓ of one period of f can also be interpreted as the Fourier coefficients z_k of the sampled signal $f(t)p(t)$ times T_s for $k = 0, \dots, n-1$.

As $f(t)p(t)$ is not band limited, we have infinitely many non zero Fourier coefficients z_k. However, due to sampling in time domain the z_k are periodic. Show that $z_{k+n} = z_k$ for all k.

Solution for Exercise 8. As $f(t)$ is T_0 periodic, it holds that $f(t) = f(t+T_0)$.

$$\begin{aligned} f(t+T_0)p(t+T_0) &= f(t) \sum_{\ell=-\infty}^{\infty} \delta(t + T_0 - \ell T_s) \\ &= f(t) \sum_{\ell=-\infty}^{\infty} \delta(t - (\ell - n)T_s). \end{aligned}$$

Substitution $u = n - \ell$ gives

$$f(t) \sum_{u=-\infty}^{\infty} \delta(t - uT_s) = f(t)p(t).$$

The T_0-periodic function $f(t)p(t)$ has fundamental frequency $\omega_0 = {}^{2\pi}/_{T_0}$. Further, it holds that

$$\omega_0 T_s = \frac{2\pi}{T_0} T_s = \frac{2\pi}{nT_s} T_s = \frac{2\pi}{n}.$$

The Fourier coefficients of $f(t)p(t)$ are therefore

$$\begin{aligned} z_k &= \frac{1}{T_0} \int_0^{T_0} f(t)p(t)e^{-jk\omega_0 t} dt \\ &= \frac{1}{T_0} \int_0^{T_0} f(t)e^{-jk\omega_0 t} \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T_s) dt \\ &= \frac{1}{T_0} \int_0^{T_0} \sum_{\ell=-\infty}^{\infty} f(t)e^{-jk\omega_0 t} \delta(t - \ell T_s) dt \\ &= \frac{1}{T_0} \int_0^{T_0} \sum_{\ell=-\infty}^{\infty} f(\ell T_s)e^{-jk\omega_0 \ell T_s} \delta(t - \ell T_s) dt \\ &= \frac{1}{T_0} \sum_{\ell=-\infty}^{\infty} f_\ell e^{-2\pi jk\ell/n} \int_0^{nT_s} \delta(t - \ell T_s) dt \end{aligned}$$

The Dirac pulse $\delta(t - \ell T_s)$ falls in the region of integration $[0, nT_s]$ if $0 \leq \ell \leq n-1$, provided we assume that the Dirac pulse is causal. Therefore

$$\int_0^{nT_s} \delta(t - \ell T_s) dt = \begin{cases} 1 & \text{if } 0 \leq \ell \leq n-1 \\ 0 & \text{else.} \end{cases}$$

This means that all summands ℓ are zero except for $\ell = 0, \dots, n-1$.

Therefore

$$\begin{aligned} z_k &= \frac{1}{nT_s}\sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n} \\ z_k T_s &= \frac{1}{n}\sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n}. \end{aligned}$$

The periodicity of the Fourier coefficients is shown as follows:

$$\begin{aligned} z_{k+n} &= \frac{1}{nT_s}\sum_{\ell=0}^{n-1} f_\ell e^{-2\pi j(k+n)\ell/n} \\ &= \frac{1}{nT_s}\sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n}\underbrace{e^{-2\pi j\ell}}_{=1} \\ &= z_k. \end{aligned}$$