Heilbronn, 21.10.2025 Prof. Dr. V. Stahl WS 25/26

Homework for Digital Signal Processing

Sheet 3

Exercise 1. Let

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

Compute the Fourier Transform $F(\omega)$ of $f(t)$.

Exercise 2. Let

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)$$

and $a \in \mathbb{R}$ with $a > 0$. Show that

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{a} F\left(\frac{\omega}{a}\right).$$

Exercise 3. Let $f \in \mathbb{R} \to \mathbb{R}$ and

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega).$$

Show that

$$f(-t) \quad \circ\!\!-\!\!\bullet \quad \overline{F(\omega)}.$$

Exercise 4. Use the frequency shift correspondence

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \hat{\omega})$$

and

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

to determine the Fourier Transform of $e^{j\hat{\omega}t}$, $\cos(\hat{\omega}t)$ and $\sin(\hat{\omega}t)$.

Exercise 5. Use the Modulation Theorem and

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

to determine the Fourier Transform of $\cos(t)$.

Use the Time Shift Theorem and

$$\sin(t) \quad = \quad \cos(t - \pi/2)$$

to obtain the Fourier Transform of $\sin(t)$. Simplify the result as much as possible using the sifting property.

Use the correspondence

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|} F(\omega/a)$$

and

$$\delta(at) \quad = \quad \frac{1}{|a|}\delta(t)$$

to derive the Fourier Transform of $\cos(\hat{\omega}t)$ from the Fourier Transform of $\cos(t)$ for arbitrary $\hat{\omega} \in \mathbb{R}$.

Exercise 6. Let $\omega = 2\pi/T$ and $u, v \in \mathbb{Z}$. Compute

$$\int_0^T e^{ju\omega t} e^{-jv\omega t} dt$$

and simplify the result as much as possible. Consider also the case $u = v$.

Exercise 7. Use the correspondences of the Fourier Transform to show that

$$f_{\hat{t}} * g \quad = \quad (f * g)_{\hat{t}}$$

where index $\hat{t}$ at a function means shifting the function by $\hat{t}$, i.e.

$$f_{\hat{t}}(t) \quad = \quad f(t - \hat{t}) \qquad \text{for all } t.$$

The proof is very short.

Exercise 8. Use the correspondences of the Fourier Transform to show that

$$f^- * g \quad = \quad (f * g^-)^-$$

where the raised minus sign means time reversal, i.e.

$$f^-(t) \quad = \quad f(-t).$$

The proof is very short.

Exercise 9. Let $f \in \mathbb{R} \to \mathbb{R}$ and

$$f^-(t) \quad = \quad f(-t)$$

for all t. The autocorrelation function f_A of f is defined by

$$f_A \quad = \quad f * f^-.$$

- Show that

$$f_A(0) \quad = \quad \int_{-\infty}^{\infty} f(t)^2 dt.$$

- Show that from $f^-(t) \circ\!\!-\!\!\bullet \overline{F(\omega)}$ and the convolution theorem it follows that

$$f_A(t) \quad \circ\!\!-\!\!\bullet \quad |F(\omega)|^2.$$

- The inverse Fourier Transform of $|F(\omega)|^2$ is

$$|F(\omega)|^2 \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega t} d\omega.$$

From this we obtain

$$f_A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 e^{j\omega t} d\omega.$$

Show that in the special case $t = 0$ we obtain Parseval's equation

$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Exercise 10. Compute the Fourier Transform of

$$\cos(2t)\sin(3t)$$

in three different ways:

- using complex exponential functions for the cosine- and sine function.
- using the modulation theorem

$$f(t)\cos(\hat{\omega})t \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}\left(F(\omega - \hat{\omega}) + F(\omega + \hat{\omega})\right).$$

- using the convolution theorem in frequency domain

$$f(t)g(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * G)(\omega).$$

Exercise 11. Let

$$f(t) = e^{j\omega t}$$

be a complex oscillation with constant angular frequency ω. Show that

$$(f * g)(t) = G(\omega)e^{j\omega t}$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau} d\tau$$

is the Fourier Transform of $g(t)$.

This means that a linear, time invariant systems answeres to an oscillation with angular frequency ω always with an oscillation with the same angular frequency ω. The oscillation is merely multiplied by a constant (time independent) factor $G(\omega)$. As $G(\omega)$ is complex this means a phase shift and an amplification of the oscillation.

Exercise 12. Figure 1 shows the Fourier Transform $F(\omega)$ of a band limited signal $f(t)$ with cut-off frequency $\hat{\omega}$.

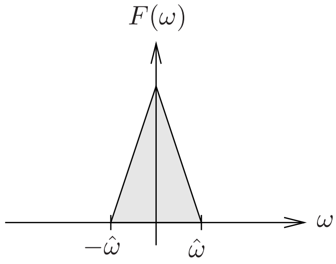

Figure 1: Fourier Transform of $f(t)$.

Let

$$f(t)\cos(\omega_0 t) \quad \circ\!\!-\!\!\bullet \quad F_{\text{mod}}(\omega).$$

where $\omega_0 > \hat{\omega}$.

- Make a sketch of $F_{\text{mod}}(\omega)$. Use the frequency shifting property of Fourier Transform.
- What might be the purpose to multiply a signal $f(t)$ before its transmission by $\cos(\omega_0 t)$?
- What is the result if the signal is multplied a second time by $\cos(\omega_0 t)$ and then convolved with a function $g(t)$ whose Fourier Transform is

$$G(\omega) = \begin{cases} 2 & \text{if } -\hat{\omega} \le \omega \le \hat{\omega} \\ 0 & \text{else?} \end{cases}$$