Homework for Digital Signal Processing

Sheet 4

Exercise 1. Let $F(\omega)$ be the Fourier Transform of $f(t)$. What do the following operations and assumptions on $f(t)$ mean for $F(\omega)$?

- $f(t)$ is multiplied by a T_s-periodic pulse train.
- $f(t)$ is T_0-periodic.
- $f(t)$ is band limited with cut-off frequency $\hat{\omega}$.

Exercise 2. Show that discrete convolution is commutative, i.e.

$$\sum_{k=-\infty}^{\infty} f_k g_{\ell-k} \quad = \quad \sum_{k=-\infty}^{\infty} g_k f_{\ell-k}.$$

Exercise 3. (Quadrature Amplitude Modulation.) The modulation theorem of Fourier Transform says that

$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega-\hat{\omega}) + F(\omega+\hat{\omega})).$$

If you consider the derivation of this theorem you can show in the same way that

$$f(t)\sin(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2j}(F(\omega-\hat{\omega}) - F(\omega+\hat{\omega})).$$

These theorems can be used to transmit *two* real signals $f(t)$ and $g(t)$ simultaneously in the same frequency band. This method is called quadrature amplitude modulation.

- The sender modulates $f(t)$ with $\cos(\hat{\omega}t)$ and $g(t)$ with $\sin(\hat{\omega}t)$ and adds both signals.
- If the receiver wants to obtain $f(t)$, he demodulates with $\cos(\hat{\omega}t)$.
- If the receiver wants to obtain $g(t)$, he demodulates with $\sin(\hat{\omega}t)$.

This can be seen as follows: The sender signal with the modulated $f(t)$ and $g(t)$ is

$$h(t) \quad = \quad f(t)\cos(\hat{\omega}t) + g(t)\sin(\hat{\omega}t).$$

- Show that

$$h(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet$$
$$\frac{1}{2}F(\omega) + \frac{1}{4}\underbrace{(F(\omega-2\hat{\omega}) + F(\omega+2\hat{\omega}) + jG(\omega-2\hat{\omega}) - jG(\omega+2\hat{\omega}))}_{\text{components in the } 2\hat{\omega} \text{ band}}$$

After lowpass filtering and multiplication with 2 we obtain $f(t)$.

- Show that

$$h(t)\sin(\hat\omega t) \quad \circ\!\!-\!\!\bullet$$
$$\frac{1}{2}G(\omega) + \frac{1}{4}\underbrace{\left(-jF(\omega-2\hat\omega) + jF(\omega+2\hat\omega) - G(\omega-2\hat\omega) - G(\omega+2\hat\omega)\right)}_{\text{components in the } 2\hat\omega \text{ band}}$$

After lowpass filtering and multiplication with 2 we obtain $g(t)$.

Exercise 4. In the lecture we derived the Fourier series of the pulse train $p(t)$:

$$\begin{aligned} p(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}, \quad \omega_s = 2\pi/T_s. \end{aligned}$$

Use this representation to compute the Fourier Transform $P(\omega)$ of $p(t)$ and show that $P(\omega)$ is also a pulse train. You may use the correspondence

$$e^{j\hat\omega t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega - \hat\omega).$$

Exercise 5. In the lecture we computed the Fourier Transform $F_p(\omega)$ of $f_p(t) = f(t)p(t)$ using the frequency shift property of the Fourier Transform:

$$f_p(t) = f(t)p(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s) = F_p(\omega).$$

An alternative way would have been to derive $F_p(\omega)$ using the convolution theorem in frequency domain:

$$f(t)p(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * P)(\omega).$$

In a previous exercise we computed $P(\omega)$. Now compute the convolution $(F * P)(\omega)$ and show that in fact

$$F_p(\omega) = \frac{1}{2\pi}(F * P)(\omega).$$

Exercise 6. (Sampling rate conversion) The formula

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \operatorname{sinc}(n - t/T_s)$$

was derived in the lecture and used to reconstruct the analog signal $f(t)$ from samples f_n.

It can also be applied to change the sampling rate. Derive a formula which takes samples f_n with sampling rate ω_s and returns new samples f'_n of the same signal $f(t)$ but with a different sampling rate ω'_s.

You may assume that $f(t)$ is band limited with cut-off frequency $\hat\omega$ and the the old and new sampling rate ω_s and ω'_s are larger than $2\hat\omega$.

Choose a suitable approximation to truncate the infinite sum to a finite sum with $2N + 1$ summands and implement a function to change the sampling rate. The function takes a finite sequence f_n of samples as well as the old and new sampling rate ω_s and ω_s' and returns a new sequence of samples f_n'.

Verify your program with a simple function like $f(t) = \cos(\hat{\omega}t)$ and $\omega_s, \omega_s' > 2\hat{\omega}$. The samples are

$$\begin{aligned} f_\ell &= f(\ell T_s) \\ &= \cos(\hat{\omega}\ell T_s). \end{aligned}$$

The samples with new sampling rate ω_s' are

$$\begin{aligned} f_\ell' &\approx f(\ell T_s') \\ &= \cos(\hat{\omega}\ell T_s'). \end{aligned}$$

Verify that the approximation is more accurate as N increases.

In order to avoid boundary effects, it is assumed that f_ℓ and f_ℓ' are infinitely long, but of course only finitely many samples of f_ℓ' are actually computed.

Exercise 7. The discrete convolution of two sequences f, g is definied by

$$(f * g)_\ell = \sum_{k=-\infty}^{\infty} g_k f_{\ell-k}.$$

If f and g are causal, this simplifies to

$$(f * g)_\ell = \sum_{k=0}^{\ell} g_k f_{\ell-k}.$$

If g has finite length N (FIR filter), the formula simplifies further to

$$(f * g)_\ell = \sum_{k=0}^{N-1} g_k f_{\ell-k}.$$

This means that for each sample ℓ the same number N of multiplications and $N - 1$ of additions have to be performed and the filter can operate in real time.

Implement a function which takes two causal sequences f, g and an integer ℓ as input and returns $(f * g)_\ell$. Test your program e.g. by choosing one signal as the Dirac pulse or a shifted Dirac pulse. The discrete Dirac pulse is defined as

$$\delta_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0. \end{cases}$$

Exercise 8. Generate a signal which consists of two cosine functions with different frequencies ω_1 and ω_2 where $\omega_1 < \omega_2$:

$$\begin{aligned} f_1(t) &= \cos(\omega_1 t) \\ f_2(t) &= \cos(\omega_2 t) \\ f(t) &= f_1(t) + f_2(t). \end{aligned}$$

Generate samples of f with sampling interval T_s:

$$f_k = f(kT_s).$$

Make sure that the samping rate $\omega_s = {}^{2\pi}/_{T_s}$ is sufficiently high, i.e.

$$\omega_s > 2\omega_2.$$

Compute samples h_ℓ of the lowpass filtered signal $h(t)$ with cutoff frequency ω_c by discrete convolution

$$h_\ell = \sum_{k=-N}^{N} g_k f_{\ell-k}$$

where

$$g_k = \hat{\omega}_c \text{sinc}(k\hat{\omega}_c), \qquad \hat{\omega}_c = \frac{2\omega_c}{\omega_s}, \qquad k = -N\ldots,N$$

are the coefficients of the truncated lowpass filter.

Choose the cutoff frequency ω_c such that

$$\omega_1 < \omega_c < \omega_2.$$

In this case the high frequency signal f_2 is filtered out and f_1 remains unchanged. Large values for N (like $N > 50$) lead to more accurate results, but take more computing time. Verify your results by comparing plots of samples of f and h. Try different values for N, ω_1, ω_2, ω_c and ω_s.

In order to avoid boundary problems you may assume that f has infinite lenght, although only finitely many samples $f_{\ell-N}, \ldots, f_{\ell+N}$ are actually needed to compute h_ℓ.