Heilbronn, 4.11.2025 Prof. Dr. V. Stahl WS 25/26

Homework for Digital Signal Processing

Sheet 5

Exercise 1. (Digital low pass filter) In the lecture the finite impulse response $\tilde{g}$ of the causal low pass filter was derived:

$$\tilde{g}_k = \begin{cases} \hat{\omega}_c \operatorname{sinc}((k - n/2)\hat{\omega}_c) & \text{for } k = 0, \ldots, n \\ 0 & \text{else.} \end{cases}$$

Here

$$\hat{\omega} = \frac{2\omega_c}{\omega_s}$$

is the normalized cutoff frequency which is between -1 and 1 due to the Sampling Theorem, ω_s is the sampling frequency, ω_c is the cutoff frequency of the low pass filter and $n/2$ is the number of samples by which the signal is delayed, which is the same as the sample where the ideal low pass filter is truncated. Larger values of n lead to a more precise filter, but cause longer delay and more computational effort.

Low pass filtering of a signal f is obtained by discrete convolution with $\tilde{g}$:

$$h_k = (f * \tilde{g})_k.$$

- Implement a function for low pass filtering. Compute the coefficients of $\tilde{g}$ only once in advance.
- Test your program e.g. by filtering a harmonic oscillation f with cutoff frequency above or below the signal frequency. In the first case the result is $h_k \approx 0$, in the second case $h_k \approx f_{k-n/2}$ for all k. Higher values of n lead to more accurate approximations.
- Another test might be to add two harmonic oscillations with different frequencies and choose the cutoff frequency in between. In this case one oscillation will vanish, the other will remain unchanged up to a delay.
- Conduct your tests also with finite lenght signals f. In this case you will observe a transient process at the beginning and at the end.
- Test also music signals so that you can hear the effect of low pass filtering.

Exercise 2. Improve your implementation of the low pass filter by using a Hamming window. (Don't forget to shift the window along with the impulse response!)

Verify with some tests, that Hamming windowing actually leads to better results, especially if the filter has only few coefficients (e.g. 10). You can use therefore an acoustic signal which consists of two harmonic oscillations and choose the cutoff frequency in between. A short impulse response leads to audible errors, which are diminished by a Hamming window.

With Matlab you can compute spectrograms of the signals and make the errors visible.

Exercise 3. Implementation of the entire system. Finish the signal processing project except for fast convolution. Replace fast convolution which you need for low pass filtering by (slow) discrete convolution in time domain.

Exercise 4. The impulse response of the ideal lowpass filter is infinite, which means that ideal lowpass filtering is not feasible in practice. Therefore we cut out a finite part of the impulse response by multiplication with a rectangular window.

- The rectangular window for $t = -\hat{t} \dots \hat{t}$ in time domain is defined by

$$r(t) \quad = \quad \begin{cases} 0 & \text{if } |t| \geq \hat{t} \\ 1 & \text{else.} \end{cases}$$

  Multiplication with a rectangular window in time domain corresponds to convolution in frequency domain. Compute the Fourier Transform $R(\omega)$. The resulting term can be simplified with the si-function.

- The error caused by windowing can be reduced by using a Hamming window instead of a rectangular window. The Hamming window for $t = -\hat{t} \dots \hat{t}$ is defined in time domain by

$$\tilde{r}(t) \quad = \quad \begin{cases} 0 & \text{if } |t| \geq \hat{t} \\ a + b\cos(\pi t/\hat{t}) & \text{else} \end{cases}$$

  where $a = 0.54$ and $b = 0.46$. Sketch the Hamming window in time domain and compute its Fourier Transform $\tilde{R}(\omega)$. The resulting term gets clearer if you use si-functions and replace $\omega\hat{t}$ by a new variable x. Compare $R(\omega)$ with $\tilde{R}(\omega)$ for different values of a and b and make plots of the functions.

  The parameters a and b are chosen such that

  - $a + b = 1$ which means that $\tilde{r}(0) = 1$.
  - $\tilde{R}(\omega) = 0$ for $\omega\hat{t} = 5\pi/2$ which means that the first side lobe in $R(\omega)$ is cancelled.

  Show that this results in $a = 25/46$ and $b = 21/46$.