Heilbronn, 11.11.2025 Prof. Dr. V. Stahl WS 25/26

# Homework for Digital Signal Processing

*Sheet 6*

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**Exercise 1.** Let $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e. a matrix whose columns are pairwise orthogonal unit vectors which means

$$\vec{a}_i \circ \vec{a}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{else} \end{cases}$$

where $\vec{a}_i$ and $\vec{a}_j$ are the $i$-th and $j$-th column vector of $A$. Show that it holds that

$$A^{\mathrm{T}}A = E$$

where $E$ is the $n \times n$ unit matix.

**Exercise 2.** Let $\vec{x}, \vec{y} \in \mathbb{C}^n$ be vectors with components

$$\begin{aligned} x_k &= e^{2\pi juk/n} \\ y_k &= e^{2\pi jvk/n} \end{aligned}$$

for $k = 0, 1, \ldots, n-1$. The complex inner product of vectors is defined as

$$\vec{x} \circ \vec{y} = \sum_{k=0}^{n-1} \overline{x_k} y_k.$$

Show that for arbitrary $u, v \in \mathbb{N}_0$ it holds that

$$\vec{x} \circ \vec{y} = \begin{cases} 0 & \text{if } u \neq v \\ n & \text{if } u = v. \end{cases}$$

Hint: Try to rewrite the inner product as

$$\vec{x} \circ \vec{y} = \sum_{k=0}^{n-1} a^k$$

for a suitable $a$ and use the formula

$$\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1}.$$

**Exercise 3.** Write a program which computes matrix $B \in \mathbb{C}^{n\times n}$ for an arbitray $n \in \mathbb{N}$ where

$$b_{k\ell} = e^{2\pi jk\ell/n}.$$

**Exercise 4.** Write a program which computes for given Fourier coefficients $\vec{z} \in \mathbb{C}^n$ the corresponding samples $\vec{f}$ by matrix vector multiplication

$$\vec{f} = B\vec{z}.$$

- The Fourier coefficients $\vec{z}$ appear in conjugate complex pairs. Make some tests that $\vec{f}$ is real if $\vec{z}$ satisfies
$$z_{n-k} = \overline{z_k} \text{ for all } k = 1, \ldots, n-1.$$
- Test that $f_k = A_0$ for all $k = 0, \ldots, n-1$, i.e. the signal is constant if
$$z_0 = A_0, \quad z_k = 0 \text{ for } k = 1, \ldots, n-1.$$
- The Fourier coefficients are given by
$$z_k = \frac{1}{2} A_k e^{j\varphi_k}, \quad k = 1, 2, \ldots, n/2-1$$
where $A_k$ is the amplitude of the $k$-th harmonic of $f(t)$ and $\varphi_k$ is its phase. Verify that the samples $\vec{f}$ are actually $m$ periods of a cosine wave with amplitude 2 if
$$z_k = \begin{cases} 1 & \text{if } k = m \text{ or } k = n-m \\ 0 & \text{else} \end{cases}$$
for $k = 1, \ldots, n/2-1$.

**Exercise 5.** Implement a function for DFT and IDFT. Test with some random vectors $\vec{z} \in \mathbb{C}^n$ and $\vec{f} \in \mathbb{R}^n$ that
$$\begin{aligned} \vec{f} &= \mathrm{IDFT}(\mathrm{DFT}(\vec{f})) \\ \vec{z} &= \mathrm{DFT}(\mathrm{IDFT}(\vec{z})). \end{aligned}$$
Due to rounding errors of floating point arithmetic small deviations may occur. Do not forget the factor $1/n$!

**Exercise 6.** Let
$$f(t) = 3 + \cos(t+1) + 2\cos(3t+2) - 5\cos(4t-1)$$
be a $T_0 = 2\pi$ periodic function. Sample $f(t)$ at 16 equidistant points in the interval $[0, 2\pi]$ and form a vector $\vec{f} \in \mathbb{R}^{16}$. What are the corresponding Fourier coefficients $\vec{z} \in \mathbb{C}^{16}$? Use the program from the previous exercise to verify that the DFT actually gives those values.

**Exercise 7.** The matrices $B$ and $B^*$ consume a lot of memory for large $n$. However, the entries of the $\ell$-th row of $B$ can be obtained from the entries of row $\ell = 1$ of $B$ by selecting every $\ell$-the element. Prove that
$$B_{\ell k} = B_{1,(k\ell) \bmod n}$$
for all $\ell, k = 0, 1, \ldots, n-1$.

Here mod is the modulo operation and
$$(k\ell) \bmod n = k\ell - un$$
where $u \in \mathbb{Z}$ is such that
$$0 \leq k\ell - un < n.$$
The same property holds for $B^*$. Implement the DFT and IDFT in a memory efficient way where only row $\ell = 1$ of $B$ is stored.

**Exercise 8.** Let

$$p(t) \quad = \quad \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T_s)$$

be a pulse train with period $T_s$.

Assume $f(t)$ is $T_0$-periodic with $T_0 = nT_s$.

Show that $f(t)p(t)$ is also $T_0$ periodic.

Compute the Fourier coefficients $z_k$ of $f(t)p(t)$ by integration and show that

$$z_k T_s \quad = \quad \underbrace{\frac{1}{n} \sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n}}_{\text{DFT of } \vec{f}}$$

where $f_\ell = f(\ell T_s)$. This means that the DFT of the samples $f_\ell$ of one period of $f$ can also be interpreted as the Fourier coefficients $z_k$ of the sampled signal $f(t)p(t)$ times $T_s$ for $k = 0, \ldots, n-1$.

As $f(t)p(t)$ is not band limited, we have infinitely many non zero Fourier coefficients $z_k$. However, due to sampling in time domain the $z_k$ are periodic. Show that $z_{k+n} = z_k$ for all $k$.