Digital Signal Processing

Prof. Dr. V. Stahl
Heilbronn University of Applied Sciences

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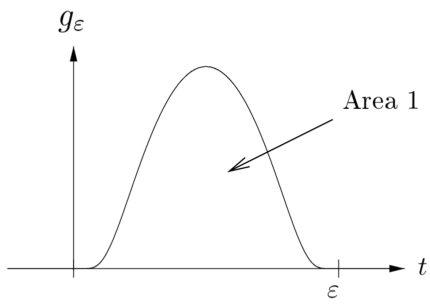
1 Foundations

1.1 Dirac Pulse

Let $\varepsilon > 0$ be a small positive number and g_ε be an impulse of width ε and area 1.



The concrete shape of g_ε is undefined, all we ask is

$$\begin{aligned} g_\varepsilon(t) &= 0 \qquad \text{for } t \notin [0,\varepsilon] \\ g_\varepsilon(t) &\geq 0 \qquad \text{for all } t \in [0,\varepsilon] \end{aligned}$$

and

$$\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1.$$

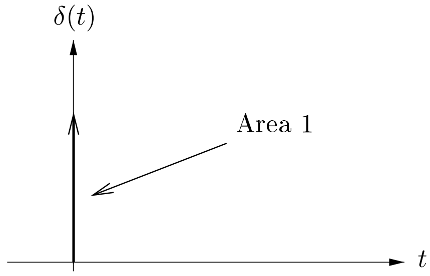
Such narrow pulses play an important role in digital signal processing, especially in the ideal case when ε goes to zero. The resulting function is called Dirac pulse and is denoted by $\delta(t)$.

From the above requirements it follows that

$$\delta(t) = 0 \text{ for all } t \neq 0$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$



Such a function can obviously not exist, hence $\delta(t)$ is called generalized function or distribution. In order to preserve the property that the area under $\delta(t)$ is one, we might require that $\delta(0)$ is infinity, but that would not help either, because zero times infinity is undefined and certainly not one. Nevertheless we sometimes write

$$\delta(t) \quad = \quad \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$

and express infinity with a vertical arrow when drawing $\delta(t)$.

The easy way to avoid such problems is to think of $\delta(t)$ as an "infinitely narrow" $g_\varepsilon(t)$. Whenever $\delta(t)$ occurs in a term, we simpliy substitute it with $g_\varepsilon(t)$ and take the limit $\varepsilon \to 0$ of the entire term, i.e. after the term was evaluated with $g_\varepsilon(t)$. This will actually do the trick:

$$\int_{-\infty}^{\infty} \delta(t)dt \quad = \quad \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{\infty} g_\varepsilon(t)dt \right) \quad = \quad \lim_{\varepsilon \to 0} 1 \quad = \quad 1.$$

In this interpretation $\delta(t)$ is nothing but a symbol similar to dx which saves us from the pain of writing limit operations all the time. For example, the derivative of $f(x)$ is defined using differential notation as

$$f'(t) \quad = \quad \frac{f(x+dx) - f(x)}{dx}$$

which is just an abbreviation for

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

This understanding of $\delta(t)$ fits its practical use. We cannot generate ideal Dirac pulses physically or experiment with them. What we can do is conduct experiments with narrow pulses like $g_\varepsilon(t)$ and observe the results when ε approaches zero. If $g_\varepsilon(t)$ represents a real physical signal like an electric voltage or a mechanical force, we may further assume that it is smooth, i.e. derivatives and integrals always exist.

A useful property of the Dirac pulse is that it is even, i.e.

$$\delta(t) \quad = \quad \delta(-t).$$

For $t \neq 0$ both sides are zero and for $t = 0$ both sides are $\delta(0)$.

A naive understanding of the Dirac pulse might lead to the conclusion, that $2\delta(t)$ is the same as $\delta(t)$. After all, two times infinity is infinity and two times zero is zero. Yet, this is not the case because the area under $2\delta(t)$ is

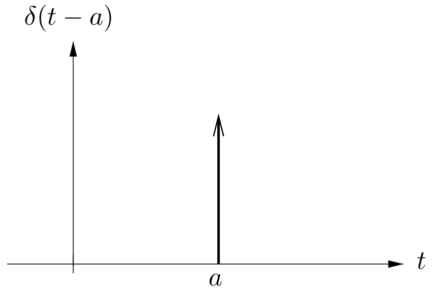
$$\int_{-\infty}^{\infty} 2\delta(t)dt \quad = \quad \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} 2g_\varepsilon(t)dt \quad = \quad 2\int_{-\infty}^{\infty} g_\varepsilon(t)dt \quad = \quad 2$$

and hence different from the area under $\delta(t)$. Therefore

$$c\delta(t) \quad \neq \quad \delta(t) \qquad \text{for } c \neq 1 \ .$$

Like any other function we can shift the Dirac pulse to an arbitrary position $t = a$ and obtain

$$\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$$



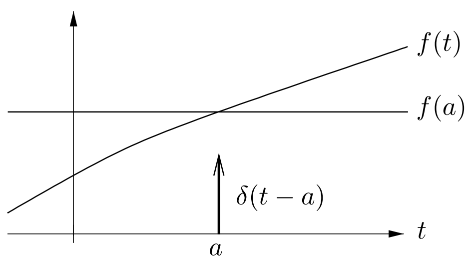
As the Dirac pulse is an even function, it holds that

$$\delta(t-a) \quad = \quad \delta(a-t).$$

The product of a function $f(t)$ and a shifted dirac pulse $\delta(t-a)$ can be simplified:

Theorem 1.1
Let $f(t)$ be continuous at $t = a$. Then

$$f(t)\delta(t-a) \quad = \quad f(a)\delta(t-a) \qquad \textit{for all } t.$$



This means that the function variable t can be replaced by constant a in the argument of f. We can convince ourselves easily that this is correct:

- For $t \neq a$ we have $t - a \neq 0$, hence $\delta(t-a) = 0$ and both sides are zero.
- For $t = a$ both sides are equal $f(a)\delta(0)$.

Remark 1.2 Actually, the above equation holds only if $f(t)$ is continuous at $t = a$. If we argue more precisely, we have

$$f(t)\delta(t-a) \quad = \quad \lim_{\varepsilon \to 0} f(t) g_\varepsilon(t-a).$$

As $g_\varepsilon(t-a) \neq 0$ only for t very close to a, we can replace $f(t)$ by its function value for t very close to a, which is

$$\lim_{t\to a} f(t).$$

But if f is not continuous at $t = a$, this limit does not exist or is different from $f(a)$.

From the above theorem we can deduce the sifting property of the Dirac pulse.

Theorem 1.3 (Sifting Property)
Let $f \in \mathbb{R} \to \mathbb{R}$. Then it holds that

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \quad = \quad f(t)$$

for all t where $f(t)$ is continuous.

The proof is as follows. With the same reasoning as above we have

$$f(\tau)\delta(t-\tau) \quad = \quad f(t)\delta(t-\tau)$$

and obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} f(t)\delta(t-\tau)d\tau \\ &= f(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau \\ &= f(t). \end{aligned}$$

1.2 Convolution

Definition 1.4 (Convolution)
The convolution $f * g$ of two functions f, g is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

Convolution is a function of functions like e.g. composition or addition of functions. Note that $(f * g)(t)$ is definined only if the improper integral converges. This is always presumed in the following.

Convolution is commutative, i.e.

$$f * g = g * f.$$

This is obtained from

$$(f * g)(t) = \int_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

with the substitution $\mu = t - \tau$ und $d\tau = -d\mu$:

$$\begin{aligned} \int_{\tau=-\infty}^{\infty} f(\tau)g(t-\tau)d\tau &= \int_{\mu=\infty}^{-\infty} -f(t-\mu)g(\mu)d\mu \\ &= \int_{\mu=-\infty}^{\infty} g(\mu)f(t-\mu)d\mu \\ &= \int_{\tau=-\infty}^{\infty} g(\tau)f(t-\tau)d\tau \\ &= (g * f)(t). \end{aligned}$$

What happens if we convolve a function f with a Dirac pulse δ? From Theorem 1.3 it follows that

$$\begin{aligned} (f * \delta)(t) &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \\ &= f(t) \end{aligned}$$

and hence

$$f * \delta = f.$$

The Dirac pulse is therefore the neutral element of convolution.

When we attach an index $\hat{t}$ at a function we mean from now on that the function is shifted by $\hat{t}$, i.e.

$$f_{\hat{t}}(t) = f(t - \hat{t}) \quad \text{for all } t.$$

Convolving a function f with a shifted Dirac pulse has the effect that the function is shifted. This follows from the above sifting theorem:

$$\begin{aligned}(f * \delta_{\hat{t}})(t) &= \int_{-\infty}^{\infty} f(\tau)\delta_{\hat{t}}(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\hat{t}-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(t-\hat{t})\delta(t-\hat{t}-\tau)d\tau \\ &= f(t-\hat{t})\int_{-\infty}^{\infty} \delta(t-\hat{t}-\tau)d\tau \\ &= f(t-\hat{t}).\end{aligned}$$

Therefore it holds that

$$f * \delta_{\hat{t}} = f_{\hat{t}}.$$

The Heaviside step function $\sigma(t)$ is defined as

$$\sigma(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

The convolution of a function f with the step function σ gives an integral of f.

$$\begin{aligned}(f * \sigma)(t) &= \int_{-\infty}^{\infty} f(\tau)\sigma(t-\tau)d\tau \\ &= \int_{-\infty}^{t} f(\tau)d\tau.\end{aligned}$$

In the last step we used that $\sigma(t-\tau)$ is zero as soon as $\tau > t$ and one otherwise.

If a function is not differentiable, we can smooth it by convolution with g_ε. If we take limit $\varepsilon \to 0$ afterwards, not much has happened. With this simple trick we can differentiate a function f, which are not differentiable in the strict sense: Smooth f by convolution with g_ε, differentiate and take limit $\varepsilon \to 0$ afterwards. This operation is called generalized derivative:

$$f'(t) = \lim_{\varepsilon \to 0}(f * g_\varepsilon)'(t).$$

To see that this actually works, we compute the derivative of $\sigma(t)$, a function which is not differentiable at $t = 0$.

$$\begin{aligned}\sigma'(t) &= \lim_{\varepsilon \to 0}(\sigma * g_\varepsilon)'(t) \\ &= \lim_{\varepsilon \to 0}\left(\int_{-\infty}^{t} g_\varepsilon(\tau)d\tau\right)' \\ &= \lim_{\varepsilon \to 0}\left([G_\varepsilon(\tau)]_{-\infty}^{t}\right)' \\ &= \lim_{\varepsilon \to 0}\left(G_\varepsilon(t) - G_\varepsilon(-\infty)\right)' \\ &= \lim_{\varepsilon \to 0} G_\varepsilon'(t) \\ &= \lim_{\varepsilon \to 0} g_\varepsilon(t) \\ &= \delta(t).\end{aligned}$$

This means that the Dirac pulse is the generalized derivative of the step function:

$$\sigma'(t) = \delta(t).$$

Convolution has many other properties, some of which are summarized in the following theorem.

Theorem 1.5 (Properties of Convolution)

Convolution has a neutral element δ.

$$f * \delta = f$$

Convolution with a shifted pulse causes shifting.

$$f * \delta_{\hat{t}} = f_{\hat{t}}$$

Convolution is commutative and associative.

$$\begin{aligned} f * g &= g * f \\ (f * g) * h &= f * (g * h) \end{aligned}$$

Convolution is a linear operator.

$$\begin{aligned} (af) * g &= a(f * g) \\ (f_1 + f_2) * g &= (f_1 * g) + (f_2 * g) \end{aligned}$$

Convolution is time invariant.

$$f_{\hat{t}} * g = (f * g)_{\hat{t}}.$$

Convolution with σ means integration.

$$(f * \sigma)(t) = \int_{-\infty}^{t} f(\tau) d\tau.$$

The proofs are not difficult and almost trivial with the help of the convolution Theorem 1.19 of the Fourier transform. Take associativity as an example

$$\begin{aligned} (f * g) * h &\circ\!\!-\!\!\bullet\ (FG)H \\ &= F(GH) \\ &\bullet\!\!-\!\!\circ\ f * (g * h). \end{aligned}$$

1.3 Fourier Series

Theorem 1.6

Let $f(t)$ be a piecewise continuous T-periodic signal. Then there exist A_k and φ_k such that

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t + \varphi_k), \qquad \omega = \frac{2\pi}{T}$$

for all t where $f(t)$ is continuous.

A T-periodic signal can be presented as a superposition of harmonic oscillations whose frequencies are entire multiples of the fundamental frequency $\omega = 2\pi/T$.

We can simplify this sum somewhat with complex numbers.

$$\begin{aligned}
f(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega t + \varphi_k) \\
&= A_0 + \sum_{k=1}^{\infty} \mathrm{re}\left(A_k e^{j(k\omega t + \varphi_k)}\right) \\
&= A_0 + \sum_{k=1}^{\infty} \mathrm{re}\left(A_k e^{j\varphi_k} e^{jk\omega t}\right) \\
&= A_0 + \frac{1}{2}\sum_{k=1}^{\infty} A_k e^{j\varphi_k} e^{jk\omega t} + \overline{A_k e^{j\varphi_k} e^{jk\omega t}} \\
&= A_0 + \frac{1}{2}\sum_{k=1}^{\infty} A_k e^{j\varphi_k} e^{jk\omega t} + A_k e^{-j\varphi_k} e^{-jk\omega t} \\
&= A_0 + \sum_{k=1}^{\infty} \frac{1}{2} A_k e^{j\varphi_k} e^{jk\omega t} + \sum_{k=1}^{\infty} \frac{1}{2} A_k e^{-j\varphi_k} e^{-jk\omega t} \\
&= \underbrace{A_0}_{z_0} + \sum_{k=1}^{\infty} \underbrace{\frac{1}{2} A_k e^{j\varphi_k}}_{z_k,\ k=1,2,\ldots} e^{jk\omega t} + \sum_{k=-1}^{-\infty} \underbrace{\frac{1}{2} A_{-k} e^{-j\varphi_{-k}}}_{z_k,\ k=-1,-2,\ldots} e^{jk\omega t} \\
&= z_0 + \sum_{k=1}^{\infty} z_k e^{jk\omega t} + \sum_{k=-\infty}^{-1} z_k e^{jk\omega t} \\
&= \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t}
\end{aligned}$$

where

$$\begin{aligned}
z_0 &= A_0 \\
z_k &= \begin{cases} 1/2 A_k e^{j\varphi_k} & \text{for } k > 0 \\ 1/2 A_{-k} e^{-j\varphi_{-k}} & \text{for } k < 0. \end{cases}
\end{aligned}$$

For example

$$\begin{aligned} z_3 &= \frac{1}{2} A_3 e^{j\varphi_3} \\ z_{-3} &= \frac{1}{2} A_3 e^{-j\varphi_3} \\ &= \overline{z_3}. \end{aligned}$$

The Fourier coefficients z_k appear in conjugate complex pairs, i.e.

$$z_{-k} = \overline{z_k}.$$

From Fourier coefficient z_k we can obtain amplitude and phase of the k-th harmonic:

$$\begin{aligned} A_0 &= z_0 \\ A_k &= 2|z_k| \\ \varphi_k &= \sphericalangle(z_k), \quad k = 1, 2, \ldots \end{aligned}$$

In order to compute the n-th Fourier coefficient z_n of function $f(t)$ we begin with the equation

$$f(t) = \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t}$$

and multiply both sides with $e^{-jn\omega t}$:

$$f(t) e^{-jn\omega t} dt = \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t} e^{-jn\omega t} dt.$$

Integration from $t = 0, \ldots, T$ on both sides gives

$$\begin{aligned} \int_0^T f(t) e^{-jn\omega t} dt &= \int_0^T \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t} e^{-jn\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} z_k \int_0^T e^{j(k-n)\omega t} dt. \end{aligned}$$

The integral can be solved as follows:

- If $k = n$ we have

$$\int_0^T e^{j(k-n)\omega t} dt = \int_0^T 1 \, dt = T.$$

- If $k \neq n$ we have

$$\begin{aligned}
\int_0^T e^{j(k-n)\omega t} dt &= \frac{1}{j(k-n)\omega}\left[e^{j(k-n)\omega t}\right]_0^T \\
&= \frac{1}{j(k-n)\omega}\left(e^{j(k-n)\omega T}-1\right) \qquad \omega = 2\pi/T \\
&= \frac{1}{j(k-n)\omega}\left(e^{2\pi j(k-n)}-1\right) \qquad k-n \in \mathbb{Z} \\
&= \frac{1}{j(k-n)\omega}(1-1) \\
&= 0
\end{aligned}$$

Hence

$$\int_0^T e^{j(k-n)\omega t} dt = \begin{cases} T & \text{if } k = n \\ 0 & \text{else.} \end{cases}$$

and therefore

$$\begin{aligned}
\int_0^T f(t) e^{-jn\omega t} dt &= \sum_{k=-\infty}^{\infty} z_k \underbrace{\int_0^T e^{j(k-n)\omega t} dt}_{T \text{ if } k=n,\ 0 \text{ else}} \\
&= z_n\, T
\end{aligned}$$

or

$$z_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega t} dt.$$

Theorem 1.7 (Fourier Series)

Let $f \in \mathbb{R} \to \mathbb{R}$ be a T-periodic funcion and

$$z_k = \frac{1}{T}\int_0^T f(t) e^{-jk\omega t} dt$$

where $\omega = 2\pi/T$. Then it holds that

$$f(t) = \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t}$$

for all t where $f(t)$ is continuous.

1.4 Fourier Transform

The decomposition into a Fourier series is only defined for T-periodic functions $f(t)$. In order to analyze also non-periodic functions in the frequency domain we take the limit $T \to \infty$. A periodic function whose period duration is infinitely long is actually a non-periodic function. This is the basic idea behind the transition from Fourier series to Fourier transform.

Definition 1.8 (Fourier Transform)
The function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

is called Fourier transform of $f(t)$*. Inversely*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

is called inverse Fourier transform of $F(\omega)$*.*

The most important properties of the Fourier transform are summarized in the appendix.

Analogy between Fourier Series and Fourier Transform.

- In the Fourier Series of a T-periodic function $f(t)$ the Fourier coefficient z_k tells us the amplitude and phase of the k-th harmonic in the frequency decomposition of $f(t)$.
- The right hand side of the inverse Fourier transform can be interpreted as a sum (integral) of oscillations $e^{j\omega t}$ over all frequencies $\omega \in \mathbb{R}$. Each oscillation is weighted by factor

  $$\frac{1}{2\pi} F(\omega) d\omega.$$

  Corresponding to the Fourier coefficent z_k, this factor tells us the amplitude and phase of the oscillation with frequency ω in the frequency decomposition of $f(t)$. As $d\omega$ is "infintely small", each single oscillation has only infinitesimal amplitude provided that $F(\omega)$ is finite.

  A measure for the energy of a signal $f(t)$ in some frequency range $[\omega_{\min}, \omega_{\max}]$ can be obtained by

  $$\frac{1}{2\pi} \int_{\omega_{\min}}^{\omega_{\max}} |F(\omega)|^2 d\omega,$$

  see Parseval's Theorem 1.34 in Section 1.4.6.

1.4.1 Example

Let $T > 0$ and

$$f(t) = \begin{cases} 1 & \text{if } -T < t < T \\ 0 & \text{else} \end{cases}$$

be a rectangular pulse. The Fourier Transform $F(\omega)$ of $f(t)$ is

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\
&= \int_{-T}^{T} e^{-j\omega t}dt \\
&= \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T}^{T} \qquad \text{if } \omega \neq 0 \\
&= -\frac{1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) \\
&= \frac{1}{j\omega}\left(e^{j\omega T} - e^{-j\omega T}\right) \\
&= \frac{1}{j\omega}2j\mathrm{im}\left(e^{j\omega T}\right) \qquad \text{as } z - \overline{z} = 2j\mathrm{im}(z) \\
&= \frac{2}{\omega}\sin(\omega T) \\
&= 2T\frac{\sin(\omega T)}{\omega T} \\
&= 2T\mathrm{si}(\omega T)
\end{aligned}$$

where the si-function is defined as

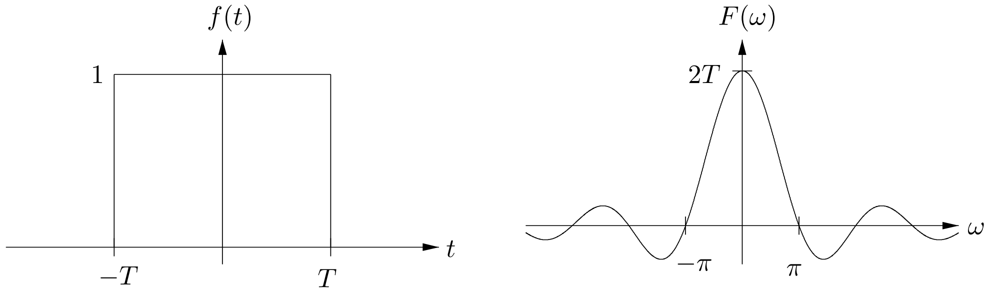
$$\mathrm{si}(x) = \begin{cases} \sin(x)/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

For $\omega = 0$ we obtain

$$F(\omega) = \int_{-\infty}^{\infty} f(t)dt = 2T = 2T\mathrm{si}(\omega T).$$

Hence

$$F(\omega) = 2T\mathrm{si}(\omega T) \quad \text{for all } \omega \in \mathbb{R}.$$



1.4.2 Properties of the Fourier Transform

The computation of a Fourier transform can be tedious because it involves the solution of an improper integral. In many cases this can be avoided by exploiting properties of the Fourier transform. With the help of these correspondences we can reduce a given function to "similar" functions whose Fourier transform has already been derived.

Theorem 1.9 (Fourier Transform is Linear)
Let

$$\begin{aligned} f(t) &\circ\!\!-\!\!\bullet\ F(\omega) \\ g(t) &\circ\!\!-\!\!\bullet\ G(\omega \end{aligned}$$

and $a \in \mathbb{C}$ be a constant. Then

$$\begin{aligned} (f+g)(t) &\circ\!\!-\!\!\bullet\ (F+G)(\omega) \\ (af)(t) &\circ\!\!-\!\!\bullet\ (aF)(\omega). \end{aligned}$$

Proof. All we have to do is use the linearity of the integral to obtain linearity of the Fourier transform.

$$\begin{aligned} (f+g)(t) \quad &\circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} (f+g)(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} (f(t)+g(t))e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \big(f(t)e^{-j\omega t} + g(t)e^{-j\omega t}\big)dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \\ &= F(\omega) + G(\omega) \\ &= (F+G)(\omega) \\ (af)(t) \quad &\circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} (af)(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} af(t)e^{-j\omega t}dt \\ &= a\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \\ &= aF(\omega) \\ &= (aF)(\omega). \ \square \end{aligned}$$

Theorem 1.10
Let

$$f(t) \circ\!\!-\!\!\bullet\ F(\omega).$$

Then

$$f(-t) \circ\!\!-\!\!\bullet\ F(-\omega).$$

Proof. With substitution

$$\tau = -t, \qquad \frac{d\tau}{dt} = -1, \qquad dt = -d\tau$$

we obtain

$$\begin{aligned} f(-t) \circ\!\!-\!\!\bullet\ & \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt \\ = & \int_{\infty}^{-\infty} f(\tau)e^{j\omega\tau}(-d\tau) \\ = & \int_{-\infty}^{\infty} f(\tau)e^{j\omega\tau}d\tau) \\ = & \int_{-\infty}^{\infty} f(\tau)e^{-j(-\omega)\tau}d\tau \\ = & F(-\omega). \ \square \end{aligned}$$

Theorem 1.11
Let

$$f(t) \circ\!\!-\!\!\bullet\ F(\omega).$$

Then

$$\overline{f(t)} \circ\!\!-\!\!\bullet\ \overline{F(-\omega)}.$$

Proof.

$$\begin{aligned} \overline{f(t)} \circ\!\!-\!\!\bullet\ & \int_{-\infty}^{\infty} \overline{f(t)}e^{-j\omega t}dt \\ = & \int_{-\infty}^{\infty} \overline{f(t)e^{j\omega t}}dt \\ = & \int_{-\infty}^{\infty} \overline{f(t)e^{j\omega t}}dt \\ = & \overline{\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt} \\ = & \overline{F(-\omega)}. \ \square \end{aligned}$$

In the special case when f is real and $f(t) = \overline{f(t)}$ we obtain

$$F(-\omega) = \overline{F(\omega)}.$$

Theorem 1.12

- *Let $f \in \mathbb{R} \to \mathbb{R}$ be a* real valued *function. Then*
$$F(-\omega) = \overline{F(\omega)}.$$
- *If $f \in \mathbb{R} \to \mathbb{R}$ is an even function then $F(\omega)$ is real.*
- *If $f \in \mathbb{R} \to \mathbb{R}$ is an odd function then $F(\omega)$ is purely imaginary.*

Proof.

- Let $f \in \mathbb{R} \to \mathbb{R}$. Then
$$\begin{aligned} F(-\omega) &= \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) \overline{e^{-j\omega t}} dt \\ &= \int_{-\infty}^{\infty} \overline{f(t) e^{-j\omega t}} dt \qquad \text{as } f(t) \text{ is real} \\ &= \overline{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt} \\ &= \overline{F(\omega)}. \end{aligned}$$
- Let $f \in \mathbb{R} \to \mathbb{R}$ be even, i.e.
$$f(-t) = f(t).$$
From Theorem 1.10 it follows that
$$F(-\omega) = F(\omega)$$
and as $f(t)$ is real we obtain
$$\overline{F(\omega)} = F(\omega).$$
Hence $F(\omega)$ is real.
- Let $f \in \mathbb{R} \to \mathbb{R}$ be odd, i.e.
$$f(-t) = -f(t).$$
From Theorem 1.10 and linearity it follows that
$$F(-\omega) = -F(\omega)$$
and as $f(t)$ is real we obtain
$$\overline{F(\omega)} = -F(\omega).$$
Hence $F(\omega)$ is purely imaginary. □

Theorem 1.13 (Time Shift)

$$f(t-\hat{t}) \quad \circ\!\!-\!\!\bullet \quad F(\omega)e^{-j\omega\hat{t}}$$

Proof. With subtitution $\tau = t - \hat{t}$ und $d\tau = dt$ we obtain

$$\begin{aligned}
f(t-\hat{t}) \quad \circ\!\!-\!\!\bullet \quad & \int_{-\infty}^{\infty} f(t-\hat{t})e^{-j\omega t}dt \\
= \quad & \int_{-\infty}^{\infty} f(\tau)e^{-j\omega(\tau+\hat{t})}d\tau \\
= \quad & \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}e^{-j\omega\hat{t}}d\tau \\
= \quad & e^{-j\omega\hat{t}}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau \\
= \quad & e^{-j\omega\hat{t}}F(\omega). \ \square
\end{aligned}$$

Theorem 1.14 (Frequency Shift)

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega-\hat{\omega})$$

Proof.

$$\begin{aligned}
f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad & \int_{-\infty}^{\infty} f(t)e^{j\hat{\omega}t}e^{-j\omega t}dt \\
= \quad & \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\hat{\omega})t}dt \\
= \quad & F(\omega-\hat{\omega}). \ \square
\end{aligned}$$

Theorem 1.15 (Modulation Theorem)

$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega-\hat{\omega})+F(\omega+\hat{\omega}))$$

Proof. With

$$\begin{aligned}
\cos(\hat{\omega}t) &= \frac{1}{2}\left(e^{j\hat{\omega}t}+e^{-j\hat{\omega}t}\right) \\
\sin(\hat{\omega}t) &= \frac{1}{2j}\left(e^{j\hat{\omega}t}-e^{-j\hat{\omega}t}\right)
\end{aligned}$$

we obtain

$$\begin{aligned}
f(t)\cos(\hat{\omega}t) &= \frac{1}{2}\left(f(t)e^{j\hat{\omega}t}+f(t)e^{-j\hat{\omega}t}\right) \\
f(t)\sin(\hat{\omega}t) &= \frac{1}{2j}\left(f(t)e^{j\hat{\omega}t}-f(t)e^{-j\hat{\omega}t}\right).
\end{aligned}$$

Using linearity and Theorem 1.14 we obtain

$$\begin{aligned} f(t)\cos(\hat{\omega}t) &\circ\!\!-\!\!\bullet \frac{1}{2}(F(\omega-\hat{\omega})+F(\omega+\hat{\omega})) \\ f(t)\sin(\hat{\omega}t) &\circ\!\!-\!\!\bullet \frac{1}{2j}(F(\omega-\hat{\omega})-F(\omega+\hat{\omega})). \ \square \end{aligned}$$

Theorem 1.16 (Time Scaling)

$$f(at) \circ\!\!-\!\!\bullet \frac{1}{|a|}F\left(\frac{\omega}{a}\right), \qquad a \neq 0$$

Proof. We apply substitution

$$\tau = at, \qquad \frac{d\tau}{dt} = a, \qquad dt = \frac{d\tau}{a}.$$

- If $a > 0$ we obtain

$$\begin{aligned} f(at) &\circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/a}\frac{1}{a}d\tau \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)e^{-j(\omega/a)\tau}d\tau \\ &= \frac{1}{a}F\left(\frac{\omega}{a}\right) \\ &= \frac{1}{|a|}F\left(\frac{\omega}{a}\right). \end{aligned}$$

- If $a < 0$ we obtain

$$\begin{aligned} f(at) &\circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt \\ &= \int_{\infty}^{-\infty} f(\tau)e^{-j\omega\tau/a}\frac{1}{a}d\tau \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} f(\tau)e^{-j(\omega/a)\tau}d\tau \\ &= -\frac{1}{a}F\left(\frac{\omega}{a}\right) \\ &= \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \qquad \text{as } a \text{ is negative. } \square \end{aligned}$$

Theorem 1.17 (Derivative in Time Domain)

$$f'(t) \circ\!\!-\!\!\bullet j\omega F(\omega)$$

Proof. According to Definition 1.8

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$

Derivative on both sides with respect to t and using its linearity gives

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)e^{j\omega t}d\omega\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\frac{d}{dt}\left(e^{j\omega t}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)e^{j\omega t}d\omega. \end{aligned}$$

The right hand side is the inverse Fourier transform of

$$j\omega F(\omega).$$

Hence

$$f'(t) \quad \circ\!\!-\!\!\bullet \quad j\omega F(\omega). \; \square$$

Theorem 1.18 (Derivative in Frequency Domain)

$$-jtf(t) \quad \circ\!\!-\!\!\bullet \quad F'(\omega)$$

Proof. Starting with the Definition of the Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

we derive both sides with respect to ω and obtain

$$\begin{aligned} F'(\omega) &= \frac{d}{d\omega}\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right) \\ &= \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}\left(e^{-j\omega t}\right)dt \\ &= \int_{-\infty}^{\infty} -jtf(t)e^{-j\omega t}dt. \end{aligned}$$

On the right hand side we find the Fourier transform of

$$-jtf(t)$$

and hence we have

$$-jtf(t) \quad \circ\!\!-\!\!\bullet \quad F'(\omega). \; \square$$

Theorem 1.19 (Convolution in Time Domain)

$$(f * g)(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)G(\omega)$$

Proof.

$$\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \\
&\circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \right) e^{-j\omega t} dt \\
&= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} f(t-\tau)g(\tau)e^{-j\omega t} d\tau dt \\
&= \int_{\tau=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t-\tau)g(\tau)e^{-j\omega t} dt d\tau \\
&= \int_{\tau=-\infty}^{\infty} g(\tau) \underbrace{\int_{t=-\infty}^{\infty} f(t-\tau)e^{-j\omega t} dt}_{\text{Fourier transform of } f(t-\tau)} d\tau \\
&= \int_{\tau=-\infty}^{\infty} g(\tau)F(\omega)e^{-j\omega\tau} d\tau \\
&= F(\omega) \int_{\tau=-\infty}^{\infty} g(\tau)e^{-j\omega\tau} d\tau \\
&= F(\omega)G(\omega).
\end{aligned}$$

From line 3 to 4 we interchanged the order of the integrals using linearity. This is admissible as long as both integrals exist (Fubini's Theorem). From line 5 to 6 we used the time shift Theorem 1.13:

$$f(t-\tau) \quad \circ\!\!-\!\!\bullet \quad F(\omega)e^{-j\omega\tau}. \ \square$$

There exists also a correspondence for convoltion in frequency doman, which we will derive in Section 1.4.5 as an example of the duality principle of the Fourier transform.

1.4.3 Fourier Transform of Standard Functions

Theorem 1.20 (Dirac Pulse)

$$\delta(t) \quad \circ\!\!-\!\!\bullet \quad 1$$

Proof. From the Sifting Theorem it follows that

$$\begin{aligned} \delta(t) \quad &\circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t) e^{0} dt \\ &= 1. \ \square \end{aligned}$$

Theorem 1.21 (Constant Function)

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

Proof.

$$\begin{aligned} 2\pi\delta(\omega) \quad &\bullet\!\!-\!\!\circ \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega \\ &= \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega \\ &= \int_{-\infty}^{\infty} \delta(\omega) e^{0} d\omega \\ &= 1. \ \square \end{aligned}$$

The sign-function is defined as follows:

$$\text{sign}(t) = \begin{cases} -1 & \text{falls } t < 0 \\ 0 & \text{falls } t = 0 \\ 1 & \text{falls } t > 0 \end{cases}$$

Theorem 1.22 (Sign Function)

$$\text{sign}(t) \quad \circ\!\!-\!\!\bullet \quad \begin{cases} 2/j\omega & \textit{if } \omega \neq 0 \\ 0 & \textit{if } \omega = 0 \end{cases}$$

Proof. It holds that

$$\text{sign}'(t) = 2\delta(t).$$

Fourier transform on both sides using derivative in time doman (Theorem 1.17) gives

$$j\omega \text{Sign}(\omega) = 2,$$

where $\text{Sign}(\omega)$ be the Fourier transform of $\text{sign}(t)$. Hence

$$\text{Sign}(\omega) = \frac{2}{j\omega} \qquad \text{for } \omega \neq 0.$$

Now about the special case $\omega = 0$, the so called DC component of $\text{sign}(t)$. From

$$\text{sign}(t) + \text{sign}(-t) \quad = \quad 0$$

it follows that

$$\text{Sign}(\omega) + \text{Sign}(-\omega) \quad = \quad 0.$$

In the special case $\omega = 0$ we have

$$\text{Sign}(0) + \text{Sign}(0) \quad = \quad 0$$

and therefore $\text{Sign}(0) = 0$. $\square$

Theorem 1.23 (Step Function)

$$\sigma(t) \quad \circ\!\!-\!\!\bullet \quad \pi\delta(\omega) + \begin{cases} 1/j\omega & \textit{if } \omega \neq 0 \\ 0 & \textit{if } \omega = 0 \end{cases}$$

Proof. First we express the step function with the sign-function.[1]

$$\sigma(t) \quad = \quad \frac{1}{2} + \frac{1}{2}\text{sign}(t).$$

Actually this is not correct for $t = 0$, because $\sigma(0) = 1$ and not $1/2$ by definition. However, From the previous theorem and $1 \circ\!\!-\!\!\bullet\ 2\pi\delta(\omega)$ it follows that

$$\sigma(t) \quad \circ\!\!-\!\!\bullet \quad \pi\delta(\omega) + \begin{cases} 1/j\omega & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0. \ \square \end{cases}$$

Now that we know the Fourier transform of the step function, we can derive a correspondence for integration in the time doman.

Theorem 1.24 (Integration in Time Doman)

$$\int_{-\infty}^{t} f(\tau)d\tau \quad \circ\!\!-\!\!\bullet \quad F(\omega)\left(\pi\delta(\omega) + \begin{cases} 1/j\omega & \textit{if } \omega \neq 0 \\ 0 & \textit{if } \omega = 0 \end{cases}\right).$$

Proof. With the Convolution Theorem 1.19 we obtain

$$\begin{aligned} \int_{-\infty}^{t} f(\tau)d\tau \quad &= \quad (f * \sigma)(t) \\ &\circ\!\!-\!\!\bullet \quad F(\omega)\left(\pi\delta(\omega) + \begin{cases} 1/j\omega & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0. \ \square \end{cases}\right) \end{aligned}$$

[1] Actually this is not correct for $t = 0$, because $\sigma(0) = 1$ and not $1/2$ by definition. However, as the Fourier transform of a function remains the same if we modify an isolated function value, we can ignore this.

1.4.4 Fourier Transform of Periodic Functions

Theorem 1.25 (Complex Oscillation)

$$e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega-\hat{\omega})$$

Proof. According to Theorem 1.21 it holds that

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

We shift in the frequency domain by $\hat{\omega}$, which leads according to Theorem 1.14 to factor $e^{j\hat{\omega}t}$ in time domain and obtain

$$e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega-\hat{\omega}). \ \square$$

Now it is easy to derive Fourier transforms of the sin- and cos-function.

Theorem 1.26 (Fourier Transform of Sine and Cosine)

$$\begin{aligned} \cos(\hat{\omega}t) \quad &\circ\!\!-\!\!\bullet \quad \pi(\delta(\omega-\hat{\omega})+\delta(\omega+\hat{\omega})) \\ \sin(\hat{\omega}t) \quad &\circ\!\!-\!\!\bullet \quad -j\pi(\delta(\omega-\hat{\omega})-\delta(\omega+\hat{\omega})) \end{aligned}$$

Proof.

$$\begin{aligned} \cos(\hat{\omega}t) \quad &= \quad \frac{1}{2}\left(e^{j\hat{\omega}t}+e^{-j\hat{\omega}t}\right) \\ &\circ\!\!-\!\!\bullet \quad \frac{1}{2}\left(2\pi\delta(\omega-\hat{\omega})+2\pi\delta(\omega+\hat{\omega})\right) \\ &= \quad \pi(\delta(\omega-\hat{\omega})+\delta(\omega+\hat{\omega})) \\ \sin(\hat{\omega}t) \quad &= \quad \frac{1}{2j}\left(e^{j\hat{\omega}t}-e^{-j\hat{\omega}t}\right) \\ &\circ\!\!-\!\!\bullet \quad \frac{1}{2j}\left(2\pi\delta(\omega-\hat{\omega})-2\pi\delta(\omega+\hat{\omega})\right) \\ &= \quad -j\pi(\delta(\omega-\hat{\omega})-\delta(\omega+\hat{\omega})). \ \square \end{aligned}$$

If $f(t)$ is a periodic function, we can decompose it into a Fourier series, see Chapter 1.3. The Fourier transform of a Fourier series is easy because thanks to linearity we merely have to apply the Fourier transform to the summands, which are simple complex oscillations. This way we can compute a Fourier transform of virtually any periodic function.

Theorem 1.27 (Fourier Transform of Fourier Series)
Let $f(t)$ be a periodic function with fundamental frequency $\hat{\omega}$ and Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} z_k e^{jk\hat{\omega}t}.$$

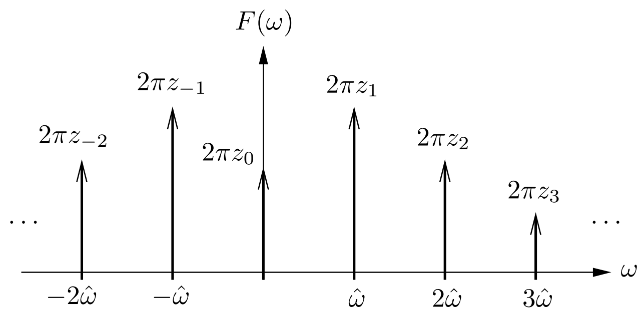
Then

$$f(t) \circ\!\!-\!\!\bullet\; 2\pi \sum_{k=-\infty}^{\infty} z_k \delta(\omega - k\hat{\omega})$$

Proof.

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z_k e^{jk\hat{\omega}t} e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} z_k \underbrace{\int_{-\infty}^{\infty} e^{jk\omega t} e^{-j\omega t} dt}_{\text{Fourier transform of } e^{jk\omega t}} \\ &= \sum_{k=-\infty}^{\infty} z_k 2\pi \delta(\omega - k\hat{\omega}) \\ &= 2\pi \sum_{k=-\infty}^{\infty} z_k \delta(\omega - k\hat{\omega}). \ \square \end{aligned}$$

The Fourier transform of a periodic function is a pulse train. The pulses are equally spaced by $\hat{\omega}$ on the frequency axis. The k-th pulse has intensity 2π times Fourier coefficient z_k. This result confirms that every oscillation component in a periodic function has a frequency which is an integer multiple of the fundamental frequency.



1.4.5 Duality Principle

It is a purely notational convention to use variable name t in time domain and ω in frequency domain. While this is useful in many situations, it can also mislead us to believe that certain properties are only true if the variables are interpreted as time or frequency. Therefore let us use a neutral variable name x instead of t or ω. We obtain

$$f(x) \quad \circ\!\!-\!\!\bullet \quad \underbrace{\int_{-\infty}^{\infty} f(u)e^{-jxu}du}_{F(x)}$$
$$f(x) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} f(u)e^{jxu}du}_{F(-x)}.$$

If

$$f(x) \quad \circ\!\!-\!\!\bullet \quad F(x)$$

then

$$f(x) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2\pi}F(-x).$$

or in one line

$$\frac{1}{2\pi}F(-x) \quad \circ\!\!-\!\!\bullet \quad f(x) \quad \circ\!\!-\!\!\bullet \quad F(x).$$

The Fourier transform and the inverse Fourier transform are identical up to a constant factor 2π and a reflection. This is the reason why every property of the Fourier transform holds with minor modifications also for its inverse and is called duality principle.

If we define $F^-(x) = F(-x)$ we can drop the function variable x and obtain

$$\frac{1}{2\pi}F^- \quad \circ\!\!-\!\!\bullet \quad f \quad \circ\!\!-\!\!\bullet \quad F.$$

Actually this is the correct notation because Fourier transform is an operator of functions f or F and not function values $f(x)$ or $F(x)$.

Now we may reintroduce function variables t and ω to make clear whether we mean time or frequency.

Theorem 1.28 (Duality Principle)
If

$$f(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)$$

then

$$f(\omega) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2\pi}F(-t).$$

Note that in the second formula we used lower case f in frequency domain and upper case F in time domain, which is against convention.

Example 1.29 It holds that

$$\underbrace{e^{j3t}}_{f(t)} \quad \circ\!\!-\!\!\bullet \quad \underbrace{2\pi\delta(\omega-3)}_{F(\omega)}$$

and inversely

$$\underbrace{e^{j3\omega}}_{f(\omega)} \quad \bullet\!\!-\!\!\circ \quad \underbrace{\delta(-t-3)}_{\frac{1}{2\pi}F(-t)} \quad = \quad \delta(t+3)$$

While this was already known, we can use the Duality Principle to derive new interesting Fourier transforms.

Example 1.30 According to Theorem 1.22 it holds that

$$\text{sign}(t) \quad \circ\!\!-\!\!\bullet \quad \begin{cases} 2/j\omega & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0 \end{cases}$$

Using duality we obtain

$$\text{sign}(\omega) \quad \bullet\!\!-\!\!\circ \quad \begin{cases} -1/j\pi t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

or

$$\begin{cases} 1/t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \circ\!\!-\!\!\bullet \quad -j\pi\text{sign}(\omega).$$

Example 1.31 According to Theorem 1.23 it holds that[2]

$$\sigma(t) \quad \circ\!\!-\!\!\bullet \quad \pi\delta(\omega) + \begin{cases} 1/j\omega & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0 \end{cases}$$

Using duality we obtain

$$\sigma(\omega) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2}\delta(t) + \begin{cases} -1/2\pi j t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

or

$$\delta(t) + \begin{cases} j/\pi t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \circ\!\!-\!\!\bullet \quad 2\sigma(\omega).$$

Going back to

$$\frac{1}{2\pi}F^{-} \quad \circ\!\!-\!\!\bullet \quad f \quad \circ\!\!-\!\!\bullet \quad F$$

we see that applying Fourier transform twice means factor 2π and reflection:

$$\frac{1}{2\pi}F^{-} \quad \circ\!\!-\!\!\bullet^{2} \quad F.$$

[2] According to footnote on page 24 we mean the modified step function, whose value is $1/2$ at zero. However, changing a function at an isolated point has no effect on its Fourier transform.

Actually we can build a sequence of Fourier transforms of functions. Let

$$f_i \quad \circ\!\!-\!\!\bullet \quad f_{i+1} \qquad \text{for } i = 1, 2, 3, \ldots.$$

Then

$$f_{i+1} = 2\pi f_{i-1}^{-}.$$

The Convolution Theorem says

$$f_i * g_i \quad \circ\!\!-\!\!\bullet \quad f_{i+1} g_{i+1}.$$

Duality says

$$\begin{aligned} f_i * g_i \quad \bullet\!\!-\!\!\circ \quad & \frac{1}{2\pi}(f_{i+1} g_{i+1})^{-} \\ = \quad & \frac{1}{2\pi} f_{i+1}^{-} g_{i+1}^{-} \\ = \quad & \frac{1}{2\pi} 2\pi f_{i-1} 2\pi g_{i-1} \\ = \quad & 2\pi f_{i-1} g_{i-1}. \end{aligned}$$

This gives us

$$f_{i-1} g_{i-1} \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi} f_i * g_i.$$

Thus we used duality to derive the Convolution Theorem in frequency domain from the Convolution Theorem in time domain.

Theorem 1.32 (Convolution in Frequency Domain)

$$(fg)(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * G)(\omega)$$

1.4.6 Autocorrelation and Parseval's Theorem

Definition 1.33 (Autocorrelation Function)
Let $f \in \mathbb{R} \to \mathbb{R}$. The autocorrelation function $f_A \in \mathbb{R} \to \mathbb{R}$ of f is defined as

$$f_A(t) = \int_{-\infty}^{\infty} f(\tau) f(\tau - t) d\tau.$$

The autocorrelation function is used to detect periodic behaviour in a noisy signal. $f_A(t)$ is the correlation of signal $f(\tau)$ with its time delayed copy $f(\tau - t)$. A high value of $f_A(t)$ is therefore an indication that $f(\tau)$ and $f(\tau - t)$ are "similar" and there is some periodicity with duration t in f.

The autocorrelation function has a maximum at $t = 0$, which is the signal energy of f:

$$f_A(0) = \int_{-\infty}^{\infty} f(\tau)^2 d\tau.$$

Autocorrelation can be expressed with convolution. Let

$$f^-(t) = f(-t)$$

then

$$f_A(t) = (f * f^-)(t).$$

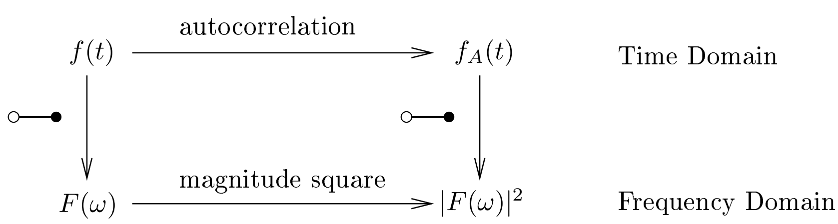
This can be seen as follows:

$$\begin{aligned}(f * f^-)(t) &= \int_{-\infty}^{\infty} f(\tau) f^-(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} f(\tau) f(-(t - \tau)) d\tau \\ &= \int_{-\infty}^{\infty} f(\tau) f(\tau - t) d\tau \\ &= f_A(t).\end{aligned}$$

The Fourier transform of f_A is easily obtained with the Convolution Theorem.

$$\begin{aligned}f_A(t) \;\circ\!\!-\!\!\bullet\; & F(\omega)\overline{F(\omega)} \\ = \; & |F(\omega)|^2,\end{aligned}$$

which is sometimes called power spectrum of f.



Inverse Fourier transform of $|F(\omega)|^2$ according to Definition 1.8 gives

$$f_A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 e^{j\omega t}d\omega.$$

In the special case $t = 0$ we obtain

$$f_A(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega.$$

Now if we recall that $f_A(0)$ is the signal energy of f, we have Parseval's Theorem.

Theorem 1.34 (Parseval's Theorem)
Let $f \in \mathbb{R} \to \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega.$$

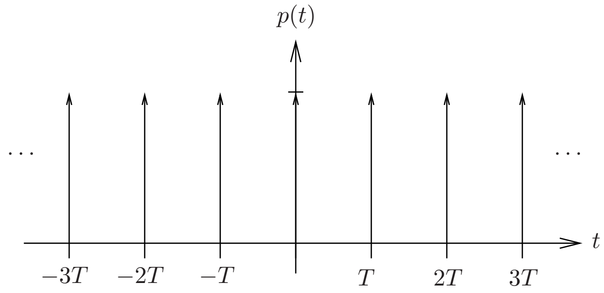
Signal energy in time and frequency domain are equal up to a constant factor 2π.

2 Sampling and Aliasing

2.1 Fourier Series of the Pulse Train

Let

$$p(t) \quad = \quad \sum_{n=-\infty}^{\infty} \delta(t - nT).$$



$p(t)$ is an infinte sequence of Dirac pulses with distance T and is therefore called pulse train or Dirac comb.

As the pulse train is T-periodic, we can decompose it into a Fourier series

$$p(t) \quad = \quad \sum_{k=-\infty}^{\infty} z_k e^{jk\omega t}, \quad \omega = \frac{2\pi}{T}.$$

In order to avoid that during the computation of Fourier coefficient z_k pulses appear at the integration limits we do not integrate from 0 to T but rather from $-{}^{T}\!/_{2}$ to ${}^{T}\!/_{2}$. If a periodic function is integrated over an entire period it is irrelevant where to start.

$$\begin{aligned}
z_k &= \frac{1}{T}\int_0^T p(t)e^{-jk\omega t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{0}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt \\
&= \frac{1}{T}.
\end{aligned}$$

The Fourier coefficients z_k of the pulse train are all equal ${}^{1}\!/_{T}$. Therefore the

Fourier series of the pulse train is

$$p(t) \quad = \quad \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega t}.$$

2.2 Sampling

Time continuous signals $f(t)$ can be processed by a digital computer only after sampling. This means that we consider only function values of $f(t)$ at discrete points in time nT_s where T_s is the sampling interval. The sampled values are

$$f_n = f(nT_s), \qquad n \in \mathbb{Z}.$$

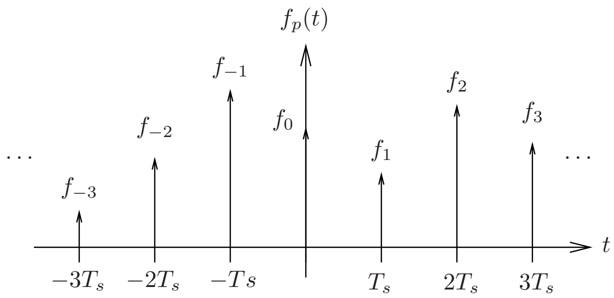
Sampling is often carried out by multiplication with a pulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$

The result is

$$\begin{aligned} f_p(t) &= f(t)p(t) \\ &= f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} f_n\delta(t - nT_s) \end{aligned}$$

The sampled signal $f_p(t)$ consits of a sequence of pulses with distance T_s where the pulse at time nT_s is multiplied by sample f_n.



Probably you wonder why sampling is done in such a complicated way: The sampled signal $f_p(t)$ is an intermediate step between analogue and discrete. On the one hand side $f_p(t)$ is still a time continuous function of t, on the other side $f_p(t)$ depends only on the samples f_n. This way we can apply all theory from analog functions and especially Fourier Transform to $f_p(t)$ and thereby transfer it to discrete signals. Further, $f(t)$ and $f_p(t)$ are actually not as different as it seems at a first glance. For small T_s the area under $f(t)$ and $f_p(t)$ is almost the same:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_p(t)dt &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_n \delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} f_n \int_{-\infty}^{\infty} \delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} f_n \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} f_n T_s \\
&\approx \frac{1}{T_s} \int_{-\infty}^{\infty} f(t)dt.
\end{aligned}$$

This does not only hold for the entire area from $-\infty$ to ∞ but also for the area over finite intervals.

2.3 Aliasing

What is the effect of sampling $f(t)$ in the frequency domain? If we use the Fourier series of the pulse train as derived in Section 2.1, we obtain

$$\begin{aligned} f_p(t) &= f(t)p(t) \\ &= f(t)\frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} f(t)e^{jk\omega_s t}. \end{aligned}$$

The Fourier Transform $F_p(\omega)$ of $f_p(t)$ is obtained from the Fourier Transform $F(\omega)$ of $f(t)$ and the correspondence

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \hat{\omega}).$$

With $\hat{\omega} = k\omega_s$ we have

$$f(t)e^{jk\omega_s t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - k\omega_s).$$

From linearity of Fourier Transform it follows that

$$\underbrace{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} f(t)e^{jk\omega_s t}}_{f_p(t)} \quad \circ\!\!-\!\!\bullet \quad \underbrace{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)}_{F_p(\omega)}.$$

This result can be interpreted as follows: $F(\omega - k\omega_s)$ is the Fourier Transform $F(\omega)$ of the original function $f(t)$ shifted by $k\omega_s$. The sum

$$\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)$$

consists of infinitely many copies of $F(\omega)$ shifted by ω_s on the frequency axis. Hence

$$F_p(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)$$

consists of infinitely many copies of $F(\omega)$ shifted by ω_s and scaled by $1/T_s$.

If the copies overlap, the overlapping parts are summed up. In that case $F(\omega)$ can no longer be obtained from $F_p(\omega)$ by cutting out one copy and scale it with T_s. This phenomenon is called Aliasing. The higher the sampling frequency ω_s is, i.e. the more samples per second we take, the farther lie the copies of $F(\omega)$ apart and the less likely overlapping occurs.

Let's keep in mind:

Sampling with T_s $\circ\!\!-\!\!\bullet$ Periodic extension with ω_s and scaling with $1/T_s$.

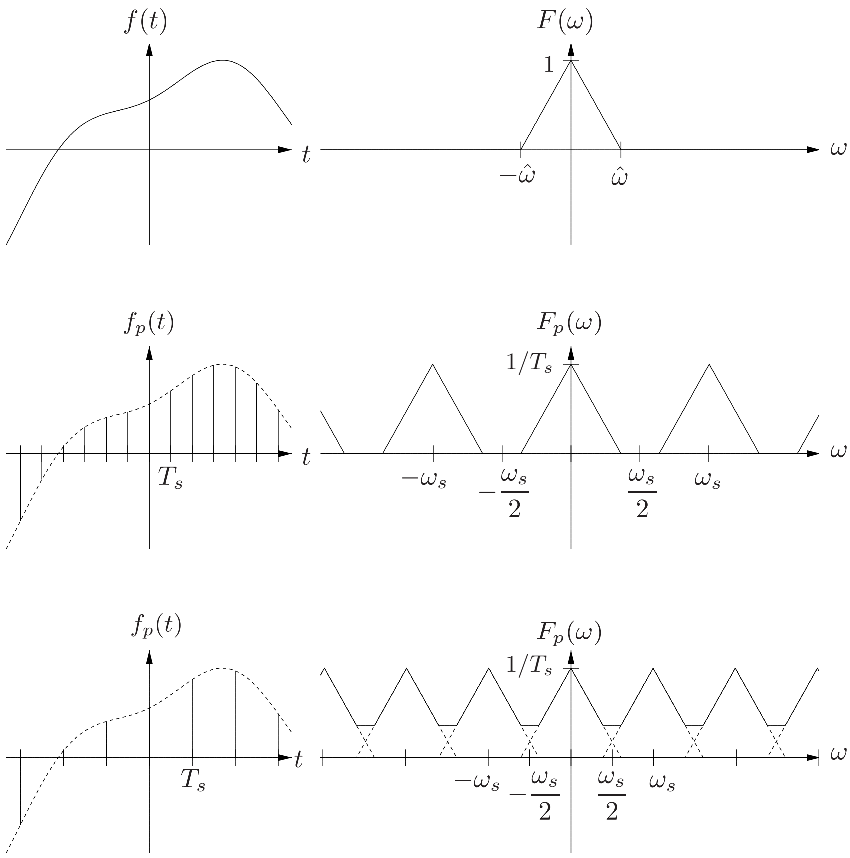


Figure 2.1: Top: Band limited signal $f(t)$ with cutoff frequency $\hat{\omega}$ and Fourier Transform $F(\omega)$. Middle: Sampled signal $f_p(t) = f(t)p(t)$. The sampling rate $\omega_s = 2\pi/T_s$ ist larger than $2\hat{\omega}$, therefore no aliasing. The copies of $F(\omega)$ do not overlap. Bottom: Sampling rate ω_s ist smaller than $2\hat{\omega}$, therefore aliasing occurs. The copies of $F(\omega)$ overlap.

2.4 Band-Limited Signals

Definition 2.1 (Band-Limited Signal)
A function $f \in \mathbb{R} \to \mathbb{R}$ is called band-limited with cutoff frequency $\hat{\omega}$ if

$$F(\omega) \quad = \quad 0 \qquad \textit{for all } \omega \textit{ with } |\omega| \geq \hat{\omega}.$$

Informally this means that $f(t)$ has no oscillation components with frequency above or equal $\hat{\omega}$.

Aliasing, i.e. an overlap in the frequency domain after sampling is avoided if the copies of $F(\omega)$ have distance more than $2\hat{\omega}$ on the frequency axis. The conditions to avoid aliasing are therefore:

- The signal $f(t)$ has to be band-limited with cutoff frequency $\hat{\omega}$.
- The sampling rate ω_s has to be sufficiently high, i.e.

$$\omega_s > 2\hat{\omega}.$$

3 Signal Reconstruction with Low Pass Filtering

After the samples of a signal have been processed in a digital computer an analog signal has to be reconstructed from the resulting, discrete samples. This step is called digital-analog conversion.

With a small modification of the theory presented in this chapter we can also build frequency selective digital filters. Filtering is done in a digital computer by discrete convolution.

3.1 Signal Reconstruction

As shown above, the Fourier Transform of $f_p(t)$ consists of infinitely many copies of $F(\omega)$ shifted by ω_s. We assume in the following that the signal $f(t)$ was band-limited with cutoff frequency $\hat{\omega}$ and the sampling frequency ω_s was high enough, i.e.

$$\omega_s > 2\hat{\omega}.$$

This means that the assumptions of the sampling theorem are satisfied and the copies of $F(\omega)$ in $F_p(\omega)$ do not overlap.

For signal reconstruction we merely have to cut out one of these copies, scale it with T_s and transform it with the inverse Fourier Transform back into time domain.

Cutting out a copy in the frequency range

$$-\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2}$$

and scaling with T_s is accomplished by multiplication with a rectangular function $G(\omega)$ in the frequency domain

$$G(\omega) = \begin{cases} T_s & \text{if } |\omega| < \omega_s/2 \\ 0 & \text{else.} \end{cases}$$

Multiplying $F_p(\omega)$ and $G(\omega)$ in the frequency domain gives the Fourier Transform $F(\omega)$ of the original time signal $f(t)$:

$$\begin{aligned} F(\omega) &= F_p(\omega)G(\omega) \\ &= \underbrace{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)}_{F_p(\omega)} G(\omega). \end{aligned}$$

The signal $f(t)$ usually comes from a sensor and is not given by a closed mathematical term. Therefore $F(\omega)$ cannot be computed analytically and the multiplication with $G(\omega)$ has to be done by convolution in time domain. According to the convolution theorem we have

$$(f_p * g)(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)G(\omega).$$

The inverse Fourier Transform $g(t)$ of $G(\omega)$ is obtained as follows:

$$\begin{aligned}
g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \\
&= \frac{T_s}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t}d\omega \\
&= \frac{T_s}{2\pi jt}\left[e^{j\omega t}\right]_{-\omega_s/2}^{\omega_s/2} \\
&= \frac{T_s}{\pi t}\,\frac{1}{2j}\left(e^{j\omega_s t/2} + e^{-j\omega_s t/2}\right) \\
&= \frac{T_s}{\pi t}\,\mathrm{im}\left(e^{j\omega_s t/2}\right) \\
&= \frac{T_s}{\pi t}\,\sin(\omega_s t/2) \\
&= \frac{T_s}{\pi t}\sin\left(\frac{\pi t}{T_s}\right) \qquad \text{da } \omega_s = \frac{2\pi}{T_s} \\
&= \mathrm{si}\left(\frac{\pi t}{T_s}\right) \\
&= \mathrm{sinc}(t/T_s).
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{si}(x) &= \begin{cases} \sin(x)/x & \text{falls } x \neq 0 \\ 1 & \text{falls } x = 0 \end{cases} \\
\mathrm{sinc}(x) &= \mathrm{si}(\pi x).
\end{aligned}$$

From the samples

$$f_k = f(kT_s)$$

we can reconstruct now the continuous signal $f(t)$ for every $t \in \mathbb{R}$ as follows by convolution in time domain.

$$\begin{aligned}
f(t) &= (f_p * g)(t) \\
&= \underbrace{\left(\sum_{k=-\infty}^{\infty} f_k\delta(t-kT_s)\right)}_{f_p(t)} * g(t) \\
&= \sum_{k=-\infty}^{\infty} f_k\left(g(t) * \delta(t-kT_s)\right) \qquad \text{(Linearity of convolution)} \\
&= \sum_{k=-\infty}^{\infty} f_k g(t-kT_s) \qquad \text{(Convolution with shifted impulse)} \\
&= \sum_{k=-\infty}^{\infty} f_k\,\mathrm{sinc}\left(\frac{t-kT_s}{T_s}\right) \\
&= \sum_{k=-\infty}^{\infty} f_k\,\mathrm{sinc}\left(t/T_s - k\right).
\end{aligned}$$

Theorem 3.1 (Sampling Theorem)
Let $f(t)$ be band-limited with cutoff frequency $\hat{\omega}$ and

$$f_k = f(kT_s), \qquad k = -\infty, \ldots, \infty, \qquad T_s = \frac{2\pi}{\omega_s}.$$

Let the sampling rate ω_s be big enough, i.e.

$$\omega_s > 2\hat{\omega}.$$

Then $f(t)$ can be reconstructed exactly from the samples f_k by

$$f(t) \quad = \quad \sum_{k=-\infty}^{\infty} f_k \operatorname{sinc}\left(t/T_s - k\right).$$

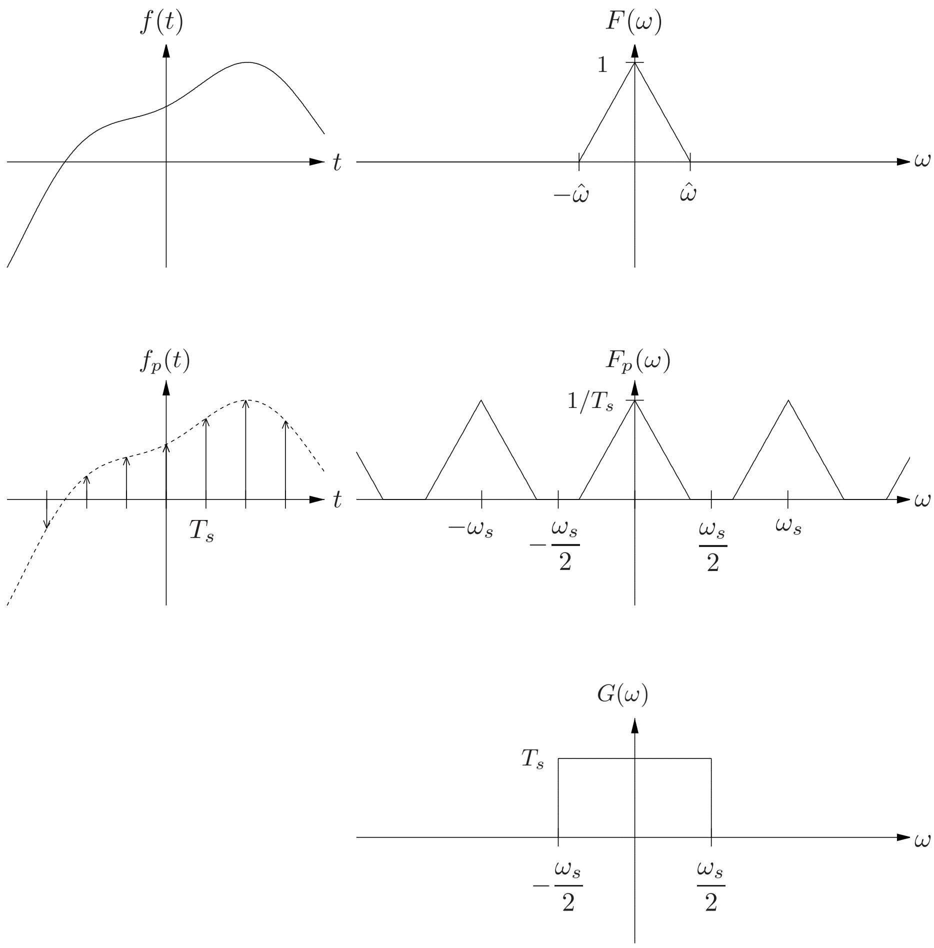


Figure 3.1: Top: Band-limited signal $f(t)$ with cutoff frequency $\hat{\omega}$ and Fourier Transform $F(\omega)$. Middle: Sampled signal $f_p(t) = f(t)p(t)$. The sampling rate $\omega_s = 2\pi/T_s$ is higher than $2\hat{\omega}$, hence the copies of $F(\omega)$ do not overlap. Bottom: Rectangular function $G(\omega)$, which is multiplied with $F_p(\omega)$ to obtain $F(\omega)$ in frequency domain.

3.2 Low Pass Filter

With the results of the previous section we will show now how to do realize a low pass filter with a given cutoff frequency ω_c. Filtering $f(t)$ means that its Fourier Transform $F(\omega)$ has to be set to zero for all ω with $|\omega| \geq \omega_c$. As according to the Sampling Theorem $f(t)$ is already band-limited with cutoff frequency $\omega_s/2$, low pass filtering makes only sense for

$$\omega_c < \frac{\omega_s}{2}.$$

In order to implement this operation in time domain on samples we merely have to adapt the rectangular function $G(\omega)$, which was used in the previous section for signal reconstruction by replacing the cut out range

$$\left[-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right]$$

with

$$[-\omega_c, \omega_c].$$

We obtain a new rectangular function for low pass filtering

$$G(\omega) = \begin{cases} T_s & \text{if } |\omega| < \omega_c \\ 0 & \text{else.} \end{cases}$$

The product

$$H(\omega) = F_p(\omega)G(\omega)$$

is now the Fourier Transform of the low pass filtered signal $h(t)$.

As in the previous section multiplication in frequency domain has to be done with convolution in time domain. As before we obtain $g(t)$ with inverse Fourier Transform:

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \\ &= \frac{T_s}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega \\ &= \frac{T_s}{2\pi jt}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right) \\ &= \frac{T_s}{\pi t}\sin(\omega_c t) \\ &= \frac{2}{\omega_s t}\sin(\omega_c t) \\ &= \frac{2\omega_c}{\omega_s}\frac{1}{\omega_c t}\sin(\omega_c t) \\ &= \frac{2\omega_c}{\omega_s}\mathrm{si}(\omega_c t) \end{aligned}$$

The low pass filtered signal $h(t)$ is obtained by convolution

$$h(t) = (f_p * g)(t).$$

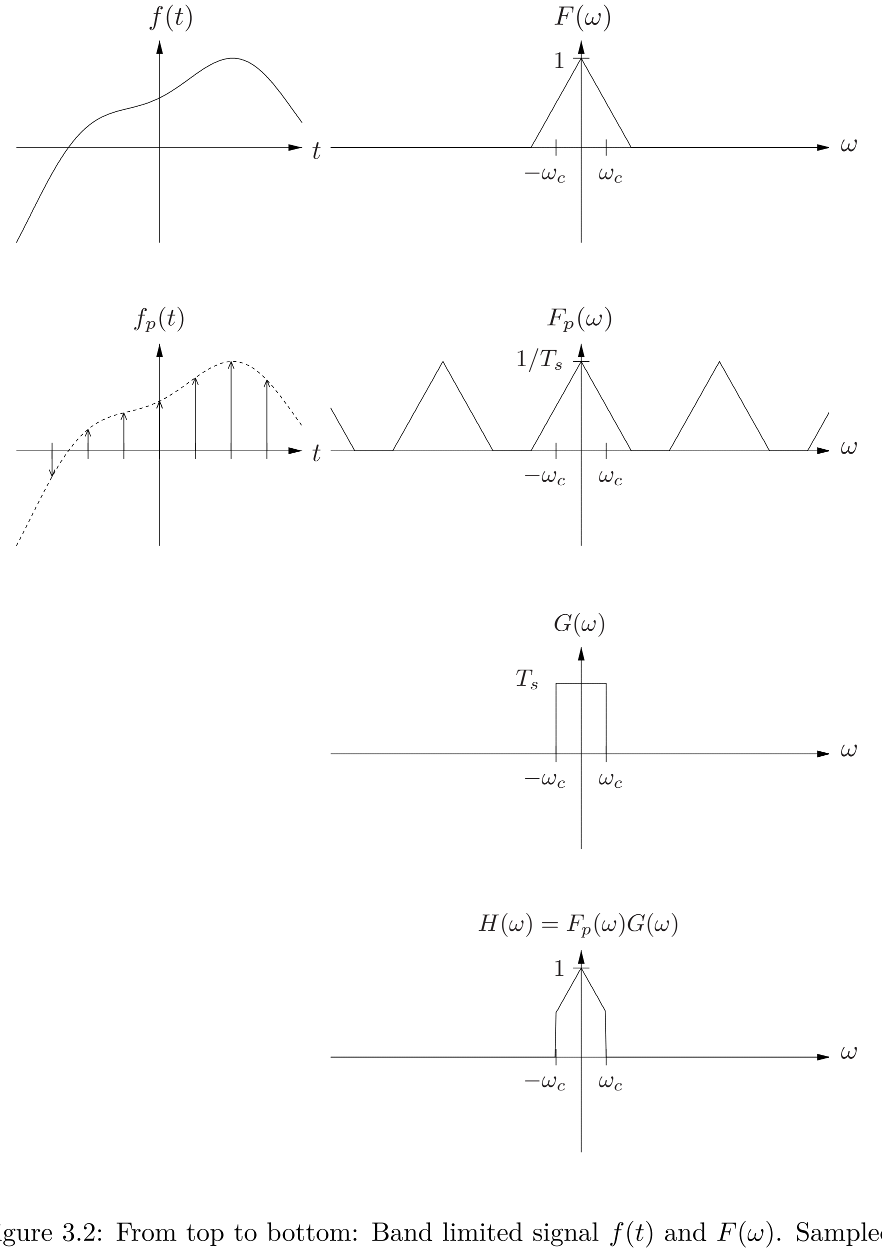


Figure 3.2: From top to bottom: Band limited signal $f(t)$ and $F(\omega)$. Sampled signal $f_p(t)$ and $F_p(\omega)$. Transfer function $G(\omega)$ of the low pass filter with cutoff frequency ω_c. Low pass filtered signal $H(\omega)$.

A low pass filter is a linear time invariant system with impulse response $g(t)$. For the practical computation there are two problems:

- The impulse response has infinite length which makes a practical computation impossible.
- The system is non-causal, i.e. $g(t) \neq 0$ for $t < 0$. Low pass filtering can therefore not be done in real time without delays.

Both problems are solved as follows:

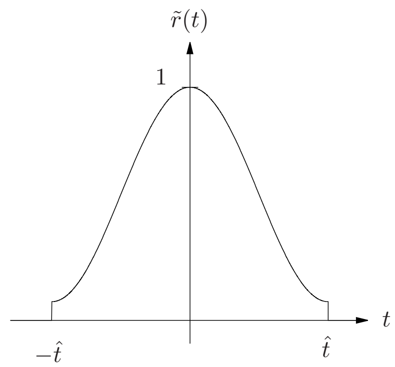
- As $g(t)$ is a si-function, its function values are small for absolutely large t and can be set to zero for $t > \hat{t}$ for some suitable $\hat{t}$. This is done in time domain by multiplication with a rectangular function $r(t)$ where

$$r(t) = \begin{cases} 0 & \text{if } |t| > \hat{t} \\ 1 & \text{else.} \end{cases}$$

  By replacing the impulse response $g(t)$ with $g(t)r(t)$ we have a system with *finite* length impulse resonse.

  Instead of cutting the impulse response "hard" at $\pm\hat{t}$ with a rectangular function $r(t)$ we can also use a soft window function. A common choice is the Hamming Window:

$$\tilde{r}(t) = \begin{cases} 0 & \text{if } |t| > \hat{t} \\ 0.54 + 0.46\cos(\pi t/\hat{t}) & \text{else.} \end{cases}$$



- The impulse response $g(t)r(t)$ is now limited to the range $\left[-\hat{t}, \hat{t}\right]$ but still not causal. This can be rectified if we shift it by $\hat{t}$. As shifting one factor in a convolution has the effect that the result of the convolution is shifted by the same amount we get merely a shift in the low pass filtered signal $h(t)$ by $\hat{t}$. A short time delay is often acceptable in signal processing applications.

Summarizing, instead of the original impulse response $g(t)$ we work with the modified causal and finite length impulse response

$$x(t) = g(t - \hat{t})r(t - \hat{t}) \qquad \text{with } x(t) = 0 \text{ for } t \notin \left[0, 2\hat{t}\right].$$

The choice of $\hat{t}$ requires a compromise: A big value of $\hat{t}$ causes small errors as only small function values of $g(t)$ are set to zero. On the other hand a big value of $\hat{t}$ causes large delays and high computation costs as shown below.

The (approximately) low pass filtered and by $\hat{t}$ delayed signal $h(t)$ is obtained in the same way as for signal reconstruction in the previous section.

$$\begin{aligned} h(t) &= (f_p * x)(t) \\ &= \left(\sum_{k=-\infty}^{\infty} f_k \delta(t - kT_s) \right) * x(t) \\ &= \sum_{k=-\infty}^{\infty} f_k \left(\delta(t - kT_s) * x(t) \right) \\ &= \sum_{k=-\infty}^{\infty} f_k x(t - kT_s). \end{aligned}$$

From this continuous signal $h(t)$ we now take samples h_ℓ.

$$\begin{aligned} h_\ell &= h(\ell T_s) \\ &= \sum_{k=-\infty}^{\infty} f_k x(\ell T_s - kT_s) \\ &= \sum_{k=-\infty}^{\infty} f_k x((\ell - k)T_s) \\ &= \sum_{k=-\infty}^{\infty} f_k x_{\ell-k} \\ &= (f * x)_\ell. \end{aligned}$$

The convolution in the last line is called *discrete* convolution of the samples f_k and x_k of the signals $f(t)$ and the impulse response $x(t)$.

Definition 3.2 (Discrete Convolution)
The discrete convolution of two sequences f_ℓ and g_ℓ is defined as

$$(f * g)_\ell = \sum_{k=-\infty}^{\infty} f_k g_{\ell-k}.$$

In order to obtain a simple representation of the filter coefficients x_k, let

$$\hat{t} = \frac{n}{2} T_s$$

for some integer n. From

$$\begin{aligned} x(t) &= g(t - \hat{t})\, r(t - \hat{t}) \\ &= \frac{2\omega_c}{\omega_s} \mathrm{si}(\omega_c t - \hat{t})\, r(t - \hat{t}) \end{aligned}$$

we obtain

$$\begin{aligned} x_k &= x(kT_s) \\ &= \frac{2\omega_c}{\omega_s}\mathrm{si}(\omega_c(kT_s - \hat{t}))\, r(kT_s - \hat{t}) \\ &= \frac{2\omega_c}{\omega_s}\mathrm{si}(\omega_c(k - n/2)T_s)\, r((k - n/2)T_s) \\ &= \frac{2\omega_c}{\omega_s}\mathrm{si}\left(\frac{2\pi\omega_c}{\omega_s}(k - n/2)\right) r_{k-n/2} \\ &= \frac{2\omega_c}{\omega_s}\mathrm{sinc}\left(\frac{2\omega_c}{\omega_s}(k - n/2)\right) r_{k-n/2} \\ &= \hat{\omega}_c\mathrm{sinc}(\hat{\omega}_c(k - n/2))\, r_{k-n/2} \end{aligned}$$

where

$$\hat{\omega}_c = \frac{2\omega_c}{\omega_s}$$

is the normalized cutoff frequency with $-1 < \hat{\omega}_c < 1$.

As the impulse response $x(t)$ has finite length due to windowing, only finitely many filter coefficients x_k are non-zero. Which ones is determined as follows: For the window function it holds that

$$r(t) = 0 \qquad \text{for } |t| > \hat{t}\ .$$

Replacing t by $(k - n/2)T_s$ on both sides we obtain

$$r((k - n/2)T_s) = 0 \qquad \text{for } \left|\left(k - \frac{n}{2}\right) T_s\right| > \frac{n}{2}T_s$$

and hence

$$r_{k-n/2} = 0 \qquad \text{for } \left|k - \frac{n}{2}\right| > \frac{n}{2}$$

or

$$r_{k-n/2} = 0 \qquad \text{for } k > n \text{ or } k < 0.$$

Therefore only filter coefficients x_k are non-zero for $k = 0, \ldots, n$.

For the rectangular window we have

$$r_{k-n/2} = 1, \quad k = 0, \ldots, n$$

and the multiplication with 1 can be ignored during the computation of the filter coefficients. For the Hamming Window we have

$$\begin{aligned}
\tilde{r}_{k-n/2} &= r((k-n/2)T_s) \\
&= 0.54 + 0.46\cos\left(\frac{\pi(k-n/2)T_s}{\hat{t}}\right) \\
&= 0.54 + 0.46\cos\left(\frac{\pi(k-n/2)T_s}{nT_s/2}\right) \\
&= 0.54 + 0.46\cos\left(\frac{\pi(2k-n)}{n}\right) \\
&= 0.54 + 0.46\cos\left(\frac{2\pi k}{n} - \pi\right) \\
&= 0.54 - 0.46\cos\left(\frac{2\pi k}{n}\right), \qquad k = 0,\dots,n.
\end{aligned}$$

Summary. The filter coefficients of the low pass filter with cutoff frequency ω_c of length n are given by

$$x_k = \hat{\omega}_c \operatorname{sinc}(\hat{\omega}_c(k-n/2)), \qquad k = 0,\dots,n$$

where $\hat{\omega}_c$ is the normalized cutoff frequency

$$\hat{\omega}_c = \frac{2\omega_c}{\omega_s}.$$

The low pass filter causes an additional delay of the signal by

$$\hat{t} = \frac{n}{2}T_s$$

or $n/2$ cycles.

As the impulse response was cut out to obtain finite length we merely obtain an approximation which is more accurate the more coefficients n the filter uses.

In addition we can improve the result by using a Hamming Window. Doing so we have to multiply x_k with

$$w_k = 0.54 - 0.46\cos\left(\frac{2\pi k}{n}\right), \qquad k = 0,\dots,n.$$

The samples h_ℓ of the low pass filtered signal are computed by discrete convolution

$$\begin{aligned}
h_\ell &= \sum_{k=-\infty}^{\infty} f_k x_{\ell-k} \\
&= \sum_{k=-\infty}^{\infty} f_{\ell-k} x_k \\
&= \sum_{k=0}^{n} f_{\ell-k} x_k.
\end{aligned}$$

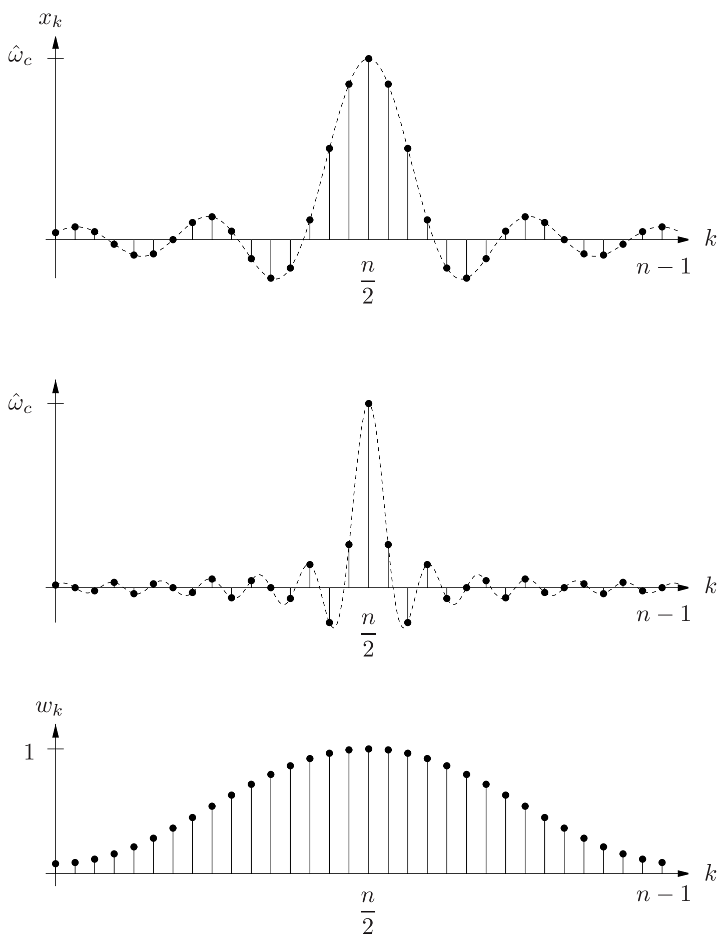


Figure 3.3: Coefficients of the low pass filter x_k of length $n = 32$ for $\hat{\omega}_c = 0.3$ (top) und $\hat{\omega}_c = 0.8$ (middle). The lower figure shows the weights w_k of the Hamming window.

3.3 Windowing*

Probably you wondered in the last section why Hamming Windowing should be an improvement over a simple rectangular window and why of all things the Hamming Window is chosen and not some other window function.

The purpose of windowing was to truncate the infinite impulse response $g(t)$ to a finite interval $[-\hat{t}, \hat{t}]$. This is accomplished by multiplication with a window function $r(t)$. What is the effect if $f(t)$ is not convolved with the ideal, infinite impulse response $g(t)$ but with a truncated version $g(t)r(t)$?

Let us compare $g(t)$ and $g(t)r(t)$ in the frequency domain. While $G(\omega)$ is an ideal rectangle and hence cuts out the desired frequency range precisely, we have to compute the Fourier Transform of $g(t)r(t)$ by convolution in the frequency domain:

$$g(t)r(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(G * R)(\omega).$$

If $R(\omega) = 2\pi\delta(\omega)$ were an impulse, windowing with $r(t)$ would not cause any distortion. However, in this case we would have $r(t) = 1$ and thus would not be limited to $[-\hat{t}, \hat{t}]$. A "good" windowing function $r(t)$ should therefore have the property that its Fourier Transform $R(\omega)$ is as close as possible to $2\pi\delta(\omega)$.

For the rectangular window

$$r(t) \quad = \quad \begin{cases} 0 & \text{if } |t| > \hat{t} \\ 1 & \text{else} \end{cases}$$

the Fourier Transform is

$$R(\omega) \quad = \quad 2\hat{t}\,\mathrm{si}(\omega\hat{t}).$$

The constant factor $2\hat{t}$ causes merely scaling. The shape of the curve is determined by the factor $\mathrm{si}(\omega\hat{t})$, which does not look like an impulse. For large values of $\omega\hat{t}$ the function value is very small. Hence, if $\hat{t}$ is large, the magnitude of $R(\omega)$ is significant only near $\omega = 0$. Near $\omega = 0$ the value of $R(\omega)$ is very large due to the multiplication with $2\hat{t}$. Thus for large $\hat{t}$ it is in fact true that $R(\omega)$ converges to an impulse. This was expected as for a large window width $\hat{t}$ only very small values of $g(t)$ are truncated.

Let us consider now the Hamming Window

$$\tilde{r}(t) \quad = \quad \begin{cases} 0 & \text{falls } |t| > \hat{t} \\ a + b\cos(\pi t/\hat{t}) & \text{else} \end{cases}$$

with $a = 0.54$ and $b = 0.46$. Its Fourier Transform is

$$\tilde{R}(\omega) \quad = \quad a\,\underbrace{2\hat{t}\,\mathrm{si}(\omega\hat{t})}_{R(\omega)} + b\hat{t}\left(\mathrm{si}(\omega\hat{t} - \pi) + \mathrm{si}(\omega\hat{t} + \pi)\right).$$

We obtain a weighted sum of $R(\omega)$ and two by $\pm\pi$ shifted si-functions. These additional summands are chosen such that the disturbing side lobes in $R(\omega)$

cancel out and a much smoother curve results, see picture 3.4. Hence the ideal transfer function $G(\omega)$ is much less distorted by a Hamming Window than it is by a rectangular window.

The figure below shows a comparison between the rectangular window (solid line) and the Hamming Window (dashed line). A signal is filtered with cutoff frequency $\omega_c = 2\pi \times 8$kHz. The impulse response of the filter is truncated at $\hat{t} = 0.2$ms. Depicted is the audible frequency range up to 20kHz.

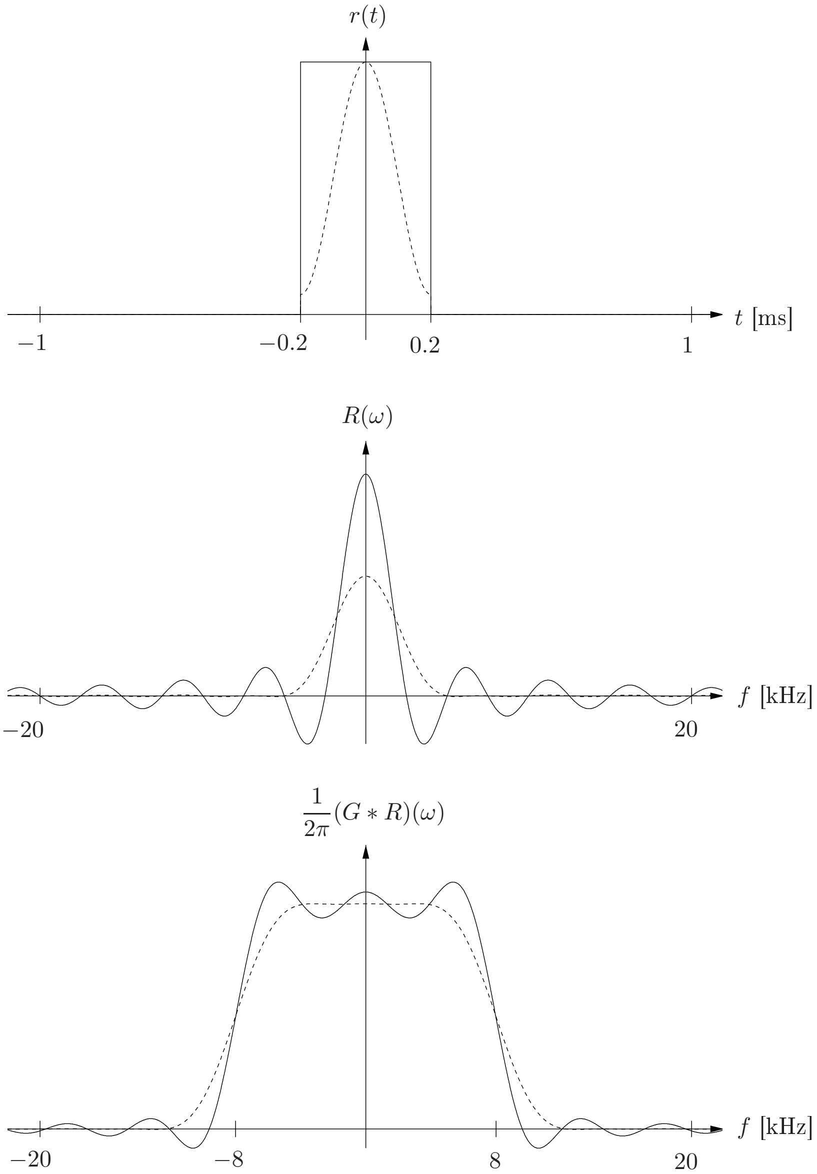


Figure 3.4: Top: Window functions $r(t)$ in time domain. Middle: Fourier Transforms $R(\omega)$ of the window functions. Bottom: The distorted transfer functions by windowing, which would ideally be a rectangle. Rectangular window: solid line, Hamming Window: dashed line.

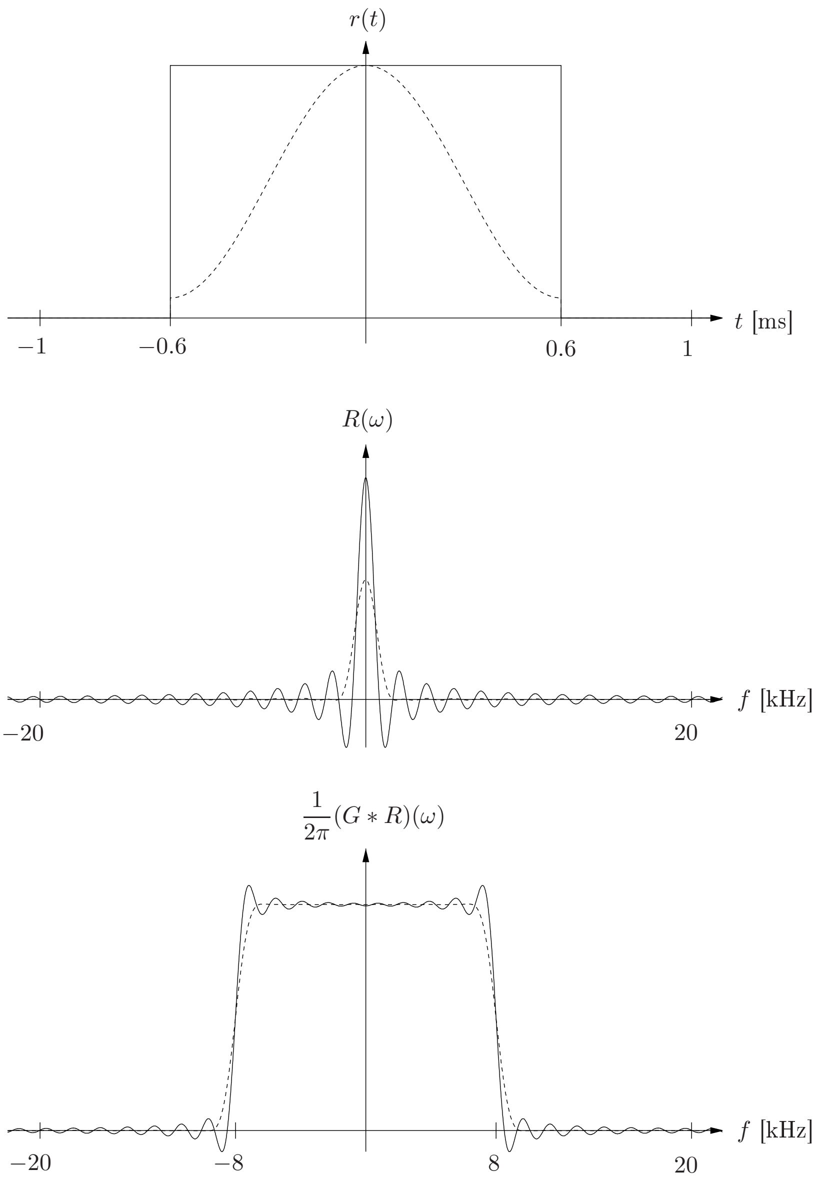


Figure 3.5: Same figure as above, except that the window width has been increased from 0.2ms to 0.6ms. $G(\omega)$ is now closer to an impulse and the transfer function is much less distorted.

3.4 Discrete Time Fourier Transform*

There is a second way to compute the Fourier Transform $F_p(\omega)$ of $f_p(t)$.

Starting with

$$f_p(t) \quad = \quad \sum_{n=-\infty}^{\infty} f_n \delta(t - nT_s)$$

we obtain from the correspondence

$$\delta(t - nT_s) \quad \circ\!\!-\!\!\bullet \quad e^{-jn\omega T_s}$$

the Fourier Transform

$$f_p(t) \quad \circ\!\!-\!\!\bullet \quad \underbrace{\sum_{n=-\infty}^{\infty} f_n e^{-jn\omega T_s}}_{F_p(\omega)}.$$

We can compute $F_p(\omega)$ either directly from the samples f_n or from $F(\omega)$:

$$\begin{aligned} F_p(\omega) &= \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s) \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-jn\omega T_s} \end{aligned}$$

From both presentations we see immediately that $F_p(\omega)$ is periodic with period ω_s, i.e.

$$F_p(\omega + \omega_s) \quad = \quad F_p(\omega) \qquad \text{for all } \omega.$$

By rescaling the frequency axis we can normalize the sampling interval to 1. For $T_s = 1$ it holds that

$$F_p(\omega) \quad = \quad \sum_{n=-\infty}^{\infty} f_n e^{-jn\omega}.$$

This transformation of a sequence f_n is called discrete time Fourier Transform (DTFT). Hence $F_p(\omega)$ ist nothing but the z-Transform of f_n for $z = e^{j\omega}$, i.e. for z on the unit circle.

From $F_p(\omega)$ we obtain by inverse Fourier Transform the original pulse train $f_p(t)$ and thus the samples f_n. However, there is also a direct way: First, we multiply both sides with $e^{jk\omega}$

$$F_p(\omega)e^{jk\omega} \quad = \quad \sum_{n=-\infty}^{\infty} f_n e^{-jn\omega} e^{jk\omega}$$

and then integrate for ω from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} F_p(\omega)e^{jk\omega} d\omega &= \int_0^{2\pi} \sum_{n=-\infty}^{\infty} f_n e^{-jn\omega} e^{jk\omega} d\omega \\ &= \sum_{n=-\infty}^{\infty} f_n \int_0^{2\pi} e^{j(k-n)\omega} d\omega \end{aligned}$$

As

$$\int_0^{2\pi} e^{j(k-n)\omega} d\omega = \begin{cases} 2\pi & \text{if } k = n \\ 0 & \text{else} \end{cases}$$

it follows that

$$\int_0^{2\pi} F_p(\omega) e^{jk\omega} d\omega = 2\pi f_k$$

or

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} F_p(\omega) e^{jk\omega} d\omega.$$

The difference to the invese Fourier Transform of $F_p(\omega)$ which would have resulted in the impulse train $f_p(t)$ is, that we integrate only from 0 to 2π and not from $-\infty$ bis ∞.

Summarizing, the DTFT and its inverse are defined as follows. The correspondence is also valid if the assumptions of the Sampling Theorem are violated. We can obtain from $F_p(\omega)$ always the samples f_k exactly but not $f(t)$ if the sampling rate was too low.

Definition 3.3 (Discrete Time Fourier Transform)

The Discrete Time Fourier Transform (DTFT) of the sequence f_k is defined by

$$F_p(\omega) = \sum_{k=-\infty}^{\infty} f_k e^{-jk\omega}.$$

Its inverse is

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} F_p(\omega) e^{jk\omega}.$$

4 Discrete Fourier Transform

4.1 Band-limited Periodic Signals

Let $f(t)$ be a T_0-periodic signal with Fourier series

$$f(t) \quad = \quad \sum_{k=-\infty}^{\infty} z_k e^{jk\omega_0 t}, \qquad \omega_0 = 2\pi/T_0.$$

Further, assume $f(t)$ is band-limited with cutoff frequency $\hat{\omega}$. This means that only finitely many Fourier Coefficients z_k are non-zero, which is shown as follows: With the correspondence

$$e^{jk\omega_0 t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega - k\omega_0)$$

we obtain the Fourier Transform $F(\omega)$ of $f(t)$:

$$F(\omega) \quad = \quad 2\pi \sum_{k=-\infty}^{\infty} z_k \delta(\omega - k\omega_0).$$

As $f(t)$ is band-limited, $F(\omega) = 0$ for $\omega \geq \hat{\omega}$ and hence

$$z_k = 0 \ \text{ if } \ k\omega_0 \geq \hat{\omega} \ \text{ i.e. if } \ k \geq \frac{\hat{\omega}}{\omega_0}.$$

The signal is sampled during one period at n equidistant places. The sampling rate is therefore

$$\omega_s \quad = \quad n\omega_0.$$

The conditions of the Sampling Theorem hold, i.e. we assume

$$\omega_s > 2\hat{\omega}.$$

It follows that

$$\frac{n}{2} > \frac{\hat{\omega}}{\omega_0}$$

and therefore

$$z_k = 0 \ \text{ if } \ k \geq \frac{n}{2}.$$

In the Fourier series only finitely many summands remain and it holds that

$$\begin{aligned} f(t) \quad &= \quad \sum_{k=-n/2+1}^{n/2-1} z_k e^{jk\omega_0 t} \\ &= \quad \sum_{k=-n/2+1}^{n/2-1} z_k e^{jk\omega_s t/n}. \end{aligned}$$

For the samples $f_\ell = f(\ell T_s)$ we have

$$\begin{aligned} f_\ell &= \sum_{k=-n/2+1}^{n/2-1} z_k e^{jk\ell\omega_s T_s/n} \\ &= \sum_{k=-n/2+1}^{n/2-1} z_k e^{2\pi jk\ell/n}. \end{aligned}$$

Cosmetics. For rewriting this formula in vector notation, the sum

$$f_\ell = \sum_{k=-n/2+1}^{n/2-1} z_k e^{2\pi jk\ell/n}$$

has to be transformed into a sum

$$f_\ell = \sum_{k=0}^{n-1} \quad \dots \qquad\qquad .$$

For this puropse we define Fourier Coefficients z_k for $k = n/2, \dots, n-1$, which do not appear in the original sum and which were zero so far as

$$\begin{aligned} z_{n/2} &= 0 \\ z_k &= z_{k-n}, \quad k = n/2+1, \dots, n-1, \end{aligned}$$

see figure 4.1. This way we can rewrite the summands with negative k.

$$\begin{aligned} \sum_{k=-n/2+1}^{-1} z_k e^{2\pi jk\ell/n} &= \sum_{k=n/2+1}^{n-1} z_{k-n} e^{2\pi j(k-n)\ell/n} \\ &= \sum_{k=n/2+1}^{n-1} z_k e^{2\pi jk\ell/n} e^{-2\pi jn\ell/n} \\ &= \sum_{k=n/2+1}^{n-1} z_k e^{2\pi jk\ell/n} \underbrace{e^{-2\pi j\ell}}_{=1} \\ &= \sum_{k=n/2+1}^{n-1} z_k e^{2\pi jk\ell/n}. \end{aligned}$$

For the entire sum it holds that

$$
\begin{aligned}
f_\ell &= \sum_{k=-n/2+1}^{n/2-1} z_k e^{2\pi jk\ell/n} \\
&= \sum_{k=-n/2+1}^{-1} z_k e^{2\pi jk\ell/n} + \sum_{k=0}^{n/2-1} z_k e^{2\pi jk\ell/n} \\
&= \sum_{k=n/2+1}^{n-1} z_k e^{2\pi jk\ell/n} + \sum_{k=0}^{n/2-1} z_k e^{2\pi jk\ell/n} \\
&= \sum_{k=0}^{n-1} z_k e^{2\pi jk\ell/n}.
\end{aligned}
$$

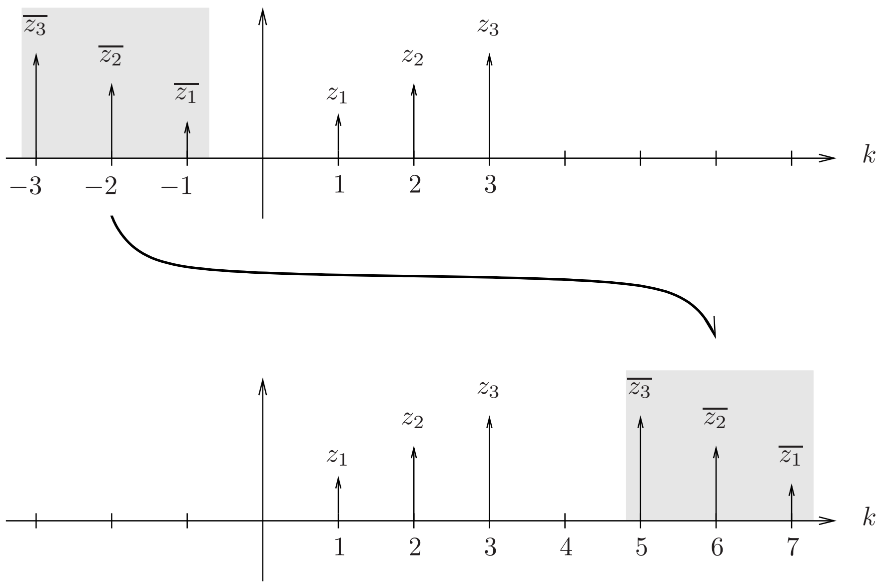


Figure 4.1: Example for $n = 8$. The coefficients z_k with $k = -1, -2, -3$ are shifted to $k = 7, 6, 5$.

4.2 Matrix Notation

In the previous section we derived a formula for the samples of a T_0-periodic signal $f(t)$. Under the assumption that the signal is band-limited and the conditions of the Sampling Theorem are satisfied it holds that

$$f_\ell \quad = \quad \sum_{k=0}^{n-1} z_k e^{2\pi jk\ell/n}.$$

We can interpret this sum as a product of a row- and a column vector:

$$f_\ell \quad = \quad \left(e^{2\pi j0\ell/n} \quad e^{2\pi j1\ell/n} \quad \dots \quad e^{2\pi j(n-1)\ell/n}\right) \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}.$$

With

$$b_{k\ell} \quad = \quad e^{2\pi jk\ell/n}$$

we obtain

$$f_\ell = (b_{0\ell} \quad b_{1\ell} \quad \dots \quad b_{n-1,\ell}) \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}.$$

If we write this formula for the n samples $f_0, \dots, f_{n-1}$ of a period one below the other we obtain a matrix by vector multiplication

$$\underbrace{\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}}_{\vec{f}} = \underbrace{\begin{pmatrix} b_{00} & b_{10} & \dots & b_{n-1,0} \\ b_{01} & b_{11} & \dots & b_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{0,n-1} & b_{1,n-1} & \dots & b_{n-1,n-1} \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}}_{\vec{z}}.$$

or simply

$$\vec{f} \quad = \quad B\vec{z}.$$

The matrix B is symmetric as

$$b_{k\ell} = e^{2\pi jk\ell/n} = e^{2\pi j\ell k/n} = b_{\ell k}.$$

Therefore we can interchange the indices of $b_{k\ell}$ arbitrarily. In order to obtain frequency components from samples we have to invert B, i.e.

$$\vec{z} \quad = \quad B^{-1}\vec{f}.$$

The computation of B^{-1} is very simple because the column vectors of B are pairwise orthogonal and have norm $\sqrt{n}$:

Theorem 4.1

Let

$$\vec{b}_u = \begin{pmatrix} b_{u0} \\ b_{u1} \\ \vdots \\ b_{u,n-1} \end{pmatrix}, \quad \vec{b}_v = \begin{pmatrix} b_{v0} \\ b_{v1} \\ \vdots \\ b_{v,n-1} \end{pmatrix}$$

be the u-th and the v-th column of B. Then it holds that

$$\vec{b}_u \circ \vec{b}_v = \begin{cases} 0 & \text{if } u \neq v \\ n & \text{else} \end{cases}$$

Proof. Let $\vec{b}_u$ and $\vec{b}_v$ as in Theorem 4.1 and

$$\begin{aligned} s &= \vec{b}_u \circ \vec{b}_v \\ &= \sum_{k=0}^{n-1} \overline{b_{uk}} b_{vk} \\ &= \sum_{k=0}^{n-1} e^{-2\pi juk/n} e^{2\pi jvk/n} \\ &= \sum_{k=0}^{n-1} e^{2\pi jk(v-u)/n}. \end{aligned}$$

- If $u = v$ then
$$e^{2\pi jk(v-u)/n} = e^0 = 1$$
and hence
$$s = \sum_{k=0}^{n-1} e^{2\pi jk(v-u)/n} = \sum_{k=0}^{n-1} 1 = n.$$

- If $u \neq v$ then
$$\begin{aligned} s &= \sum_{k=0}^{n-1} e^{2\pi jk(v-u)/n} \\ &= \sum_{k=0}^{n-1} \underbrace{\left(e^{2\pi j(v-u)/n}\right)^k}_{a} \\ &= \sum_{k=0}^{n-1} a^k. \end{aligned}$$
Multiplication with a gives
$$as = a \sum_{k=0}^{n-1} a^k = \sum_{k=1}^{n} a^k$$

and therefore

$$\begin{aligned} as - s &= \sum_{k=1}^{n} a^k - \sum_{k=0}^{n-1} a^k = a^n - 1 \\ s(a-1) &= a^n - 1 \\ s &= (a^n - 1)/(a-1). \end{aligned}$$

However, as

$$a^n = e^{2\pi j(v-u)} = 1$$

it follows that

$$s = 0.$$

We conclude from this that

$$B^* B = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix} = nE$$

where E is the $n \times n$ unit matrix and B^* the adjoint matrix of B, i.e. the matrix which is obtained by transposition of B and the complex conjugate of all components. Multiplication with B^{-1} from the right and division by n gives

$$\frac{1}{n} B^* = B^{-1}.$$

The inverse of B is obtained easily with this formula and we obtain

$$\begin{aligned} \vec{f} &= B\vec{z} \\ \vec{z} &= \frac{1}{n} B^* \vec{f}. \end{aligned}$$

From the samples f_ℓ we obtain by a simple matrix-vector multiplication the amplitudes and phases z_k of the oscillations in $f(t)$. This operation is called Discrete Fourier Transform (DFT). The inverse operatione, i.e. the computation of the samples f_ℓ from the complex Fourier Coefficients z_k is called Inverse Discrete Fourier Transform (IDFT).

Definition 4.2 (Discrete Fourier Transform)
The Discrete Fourier Transform DFT $\in \mathbb{C}^n \to \mathbb{C}^n$ *is defined by*

$$DFT(\vec{f}) = \frac{1}{n} B^* \vec{f}.$$

The Inverse Discrete Fourier Transform[a] IDFT $\in \mathbb{C}^n \to \mathbb{C}^n$ *ist defined by*

$$IDFT(\vec{z}) = B\vec{z}.$$

The matrix $B \in \mathbb{C}^{n\times n}$ *is given by*

$$b_{k\ell} = e^{2\pi jk\ell/n}, \qquad k, \ell = 0, \dots, n-1.$$

[a] In some implementations the factor $1/n$ appears in the IDFT instead of the DFT. As this is only a scaling factor, it has no influence on the essential properties of the DFT.

The formulas can be written component wise as

$$
\begin{aligned}
f_\ell &= \sum_{k=0}^{n-1} z_k e^{2\pi jk\ell/n} && \ell = 0, \ldots, n-1 \\
z_k &= \frac{1}{n}\sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n} && k = 0, \ldots, n-1.
\end{aligned}
$$

4.3 Examples

Example 4.3 Figure 4.2 shows one period of the T_0-periodic function

$$f(t) = \cos(2\omega_0 t + \pi) + 0.2\cos(3\omega_0 t) + 0.4\cos(6\omega_0 t)$$

with $\omega_0 = 2\pi/T_0$. We have a superposition of second, third and sixth harmonic. Ths signal is sampled during one period at $n = 16$ equidistant places. The sampling rate is therefore

$$\omega_s = 16\omega_0.$$

As the highes frequency in the signal is only $6\omega_0$ and therefore smaller than $\omega_s/2 = 8\omega_0$, the conditions of the Sampling Theorem are satisfied. From the samples f_ℓ, $\ell = 0, \ldots, 15$ the Fourier Coefficients z_k are computed with the DFT. From

$$\begin{aligned} z_0 &= A_0 \\ z_k &= \frac{1}{2} A_k e^{j\varphi_k}, \quad k = 1, 2, \ldots, 7 \end{aligned}$$

the amplitudes and phases of the k-th harmonic are obtained by

$$\begin{aligned} A_0 &= |z_0| \\ A_k &= 2|z_k|, \end{aligned}$$

see Figure 4.2 lower part.

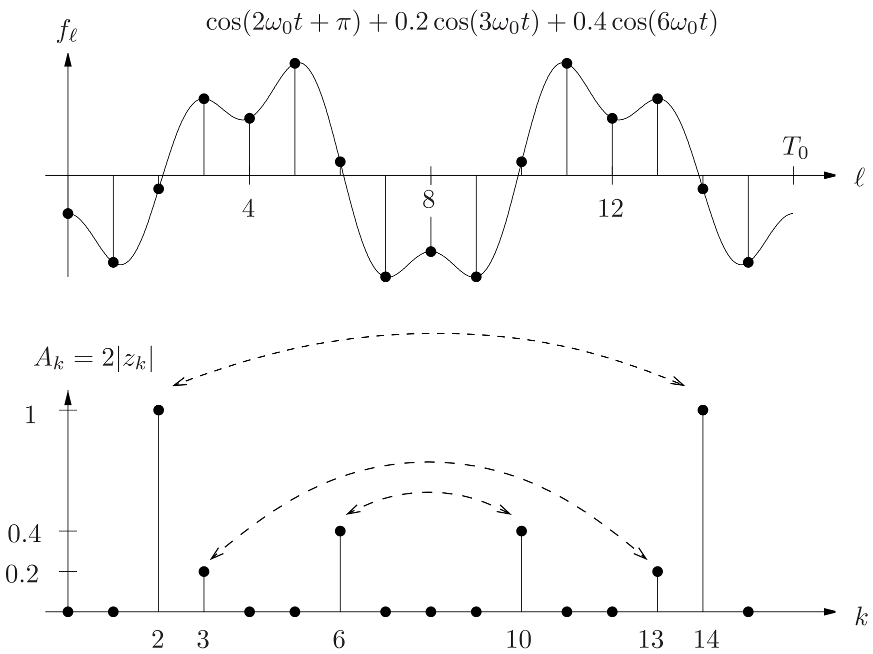


Figure 4.2: Discrete Fourier Transform for $n = 16$.

Example 4.4 The same signal as in the example above is now sampled at twice as many places, i.e. with sampling rate abgetastet, d.h. die Abtastfrequenz ist nun

$$\omega_s = 32\omega_0,$$

see Figure 4.3 upper part. Discrete Fourier Transform gives basically the sample values (Figure 4.3 lower part). Obviously there is more space for higher harmonics as in the example above. In fact the conditions of the Sampling Theorem would still be satisfied up to the 15-th harmonic while the limit above was the 7-th harmonic.

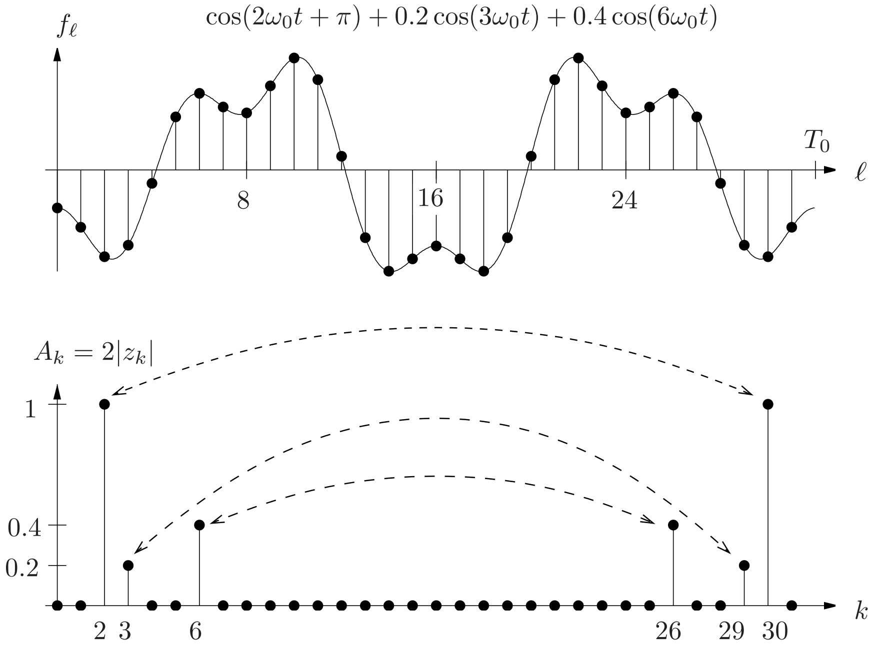


Figure 4.3: Discrete Fourier Transform for $n = 32$.

Example 4.5 A square wave contains oscillations of arbitrarily high frequency. It is therefore definitely not band-limited. Nevertheless we sample it and compute the DFT of the samples, see Figure 4.4. If one reconstructs the analog signal from the Fourier Coefficients z_k using the formulas

$$\begin{aligned} f(t) &= A_0 + \sum_{k=1}^{n/2-1} A_k \cos(k\omega t + \varphi_k) \\ z_0 &= A_0 \\ z_k &= \frac{1}{2} A_k e^{j\varphi_k}, \quad k = 1, 2, \ldots, n/2 - 1 \end{aligned}$$

only an approximation is obtained. The reason is the violation of the Sampling Theorem. All frequencies above or equal half the sampling rate lead to aliasing. However, the samples f_ℓ themselves can be reconstructed exactly from the Fourier Coefficients by

$$\vec{f} = B\vec{z}.$$

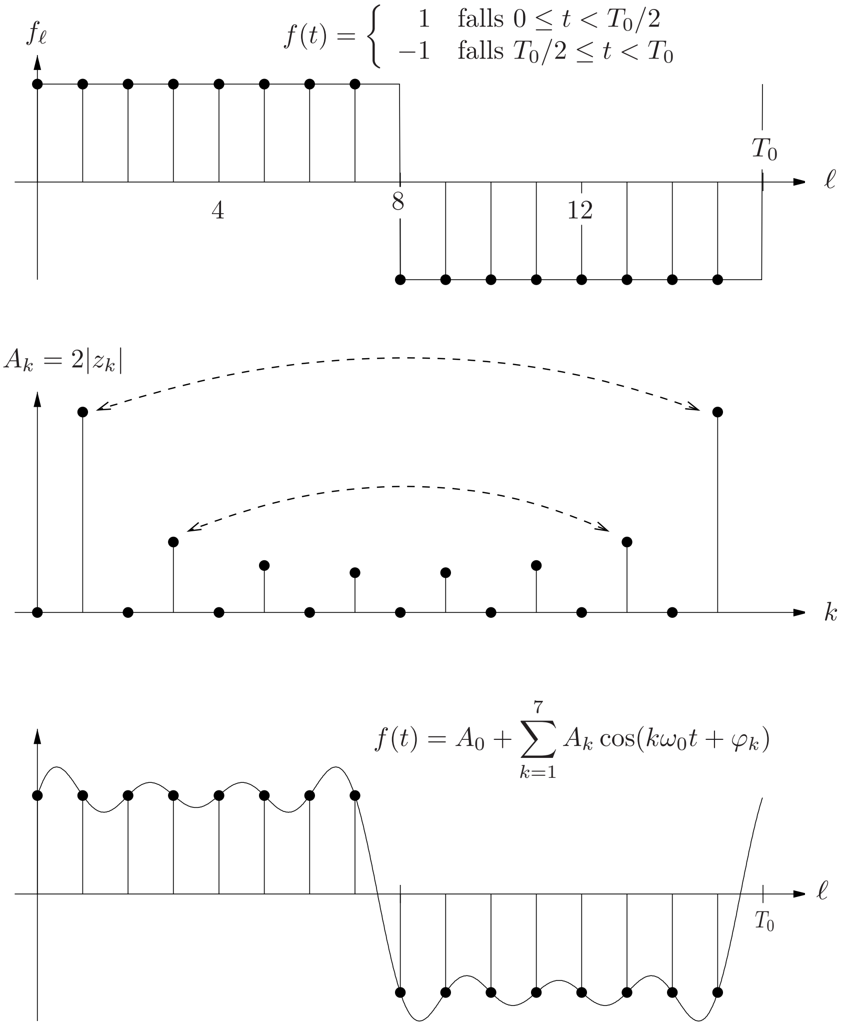


Figure 4.4: Discrete Fourier Transform for $n = 16$.

5 Fast Fourier Transform

In Definition 4.2 the DFT of a signal vector $\vec{f} \in \mathbb{R}^n$ was defined as

$$\vec{z} \;=\; \frac{1}{n} B^* \vec{f}.$$

From the Fourier Coefficients z_k we can reconstruct the original samples f_k by

$$\vec{f} \;=\; B\vec{z}.$$

This operation is called Inverse Discrete Fourier Transform (IDFT). The matrix B has the entries

$$b_{k\ell} \;=\; e^{2\pi jk\ell/n}, \quad k, \ell = 0, 1, \ldots, n-1.$$

As $b_{k\ell} = b_{\ell k}$ it holds that B is symmetric and we need not distinguish between row and column index. For $n = 4$ we have e.g.

$$B \;=\; \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}.$$

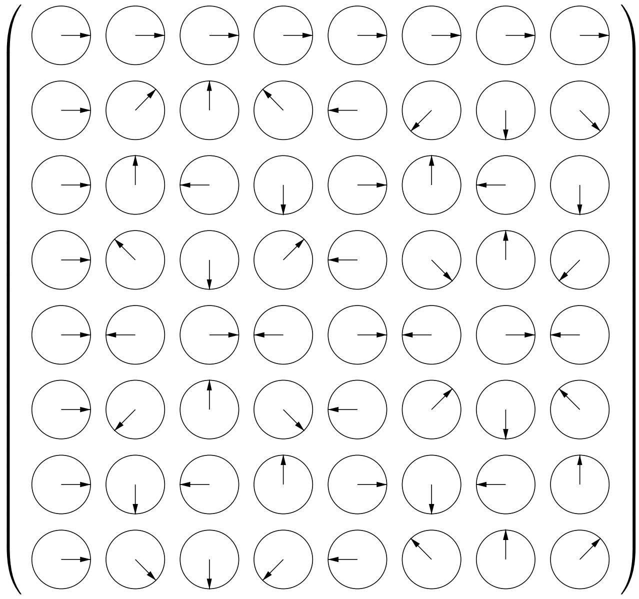
If we look at the rows of B and think of each entry as a rotating phasor we observe that

- the phasor in the first row does not rotate at all
- the phasor in the second row rotates with 90° steps
- the phasor in the third row rotates with 180° steps
- the phasor in the third row rotates with 270° steps

counter clock wise. In general the phasor in the k-th row rotates with $2\pi k/n$ radians per step in counter clock direction, see Figure 5.1 upper part. The matrix B^* is identical to B except that all entries are conjucate complex. (As B is symmetric, transposition has no effect.) In Figure 5.1 lower part this is reflected by the *clock wise* rotation of the phasors.

The computation of the DFT with matrix by vector multiplication costs n^2 complex multiplications for the evaluation of $B^*\vec{f}$. Using the regular structure of B^*, the multiplication can be done much more efficiently. This is particularly the case if n is a power of two. The algorithm for doing this is called Fast Fourier Transform (FFT).

$B =$



$B^* =$

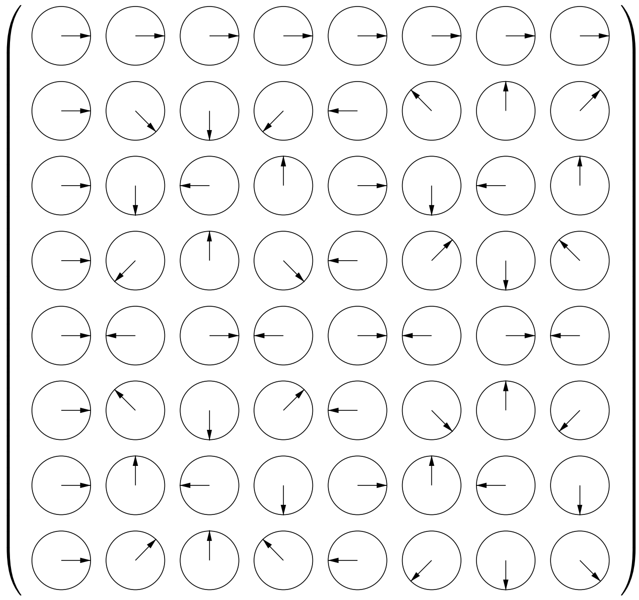


Figure 5.1: Matrix B and B^* for $n = 8$.

Simplification. In order to simplify representation we omit the cator $1/n$ temporarily in the definition of the DFT. The modified DFT in this way of a vector $\vec{f} \in \mathbb{R}^n$ is the vector $\vec{z} \in \mathbb{C}^n$ where

$$\vec{z} \quad = \quad B^* \vec{f}$$

or

$$z_k \quad = \quad (B^* \vec{f})_k \;=\; \sum_{\ell=0}^{n-1} f_\ell \, e^{-2\pi j k \ell / n}, \qquad k = 0, 1, \ldots, n-1.$$

Computational Costs. The basic idea of the FFT is to split the computation of a DFT of order n into two DFTs of order $n/2$. As the costs for matrix by vector multiplication is quadratic, computing time is reduced:

- Costs for an n-DFT:

$$n^2 \quad \text{multiplications.}$$

- Costs for two $n/2$-DFTs:

$$2\left(\frac{n}{2}\right)^2 \;=\; \frac{n^2}{2} \quad \text{multiplications.}$$

However, as shown below, the decomposition causes overhead of $n/2$ multiplications. Nevertheless total computing time is reduced, as

$$\frac{n^2}{2} + \frac{n}{2} < n^2.$$

Obviously each of the two $n/2$ DFTs can be divided up into two $n/4$-DFTs and so on. If n is a power of two this decomposition can be repeated $\mathrm{ld}(n)$ times. Finally one has DFTs of order 1 which do not involve any computation. Each of the $\mathrm{ld}(n)$ causes overhead of $n/2$ multiplicaionts, so that the total effort for the FFT is

$$\frac{n}{2}\mathrm{ld}(n)$$

multiplications. For $n = 1024$ we obtain

$$n^2 \approx 10^6, \qquad \frac{n}{2}\mathrm{ld}(n) \approx 500,$$

i.e. FFT is faster than DFT by factor 200.

Formulas. The sum for the computation of z_k is decomposed into two sub sums where the first sub sum comprises of the even samples of $\vec{f}$ and the second of the odd ones.

$$\begin{aligned} z_k &= \sum_{\ell=0}^{n-1} f_\ell e^{-2\pi jk\ell/n} \\ &= \sum_{\ell=0,2,4,\ldots} f_\ell e^{-2\pi jk\ell/n} + \sum_{\ell=1,3,5,\ldots} f_\ell e^{-2\pi jk\ell/n}. \end{aligned}$$

The summation index ℓ can iterate continuously from 0 to $n/2-1$ if ℓ is replaced by 2ℓ in the first sum and by $2\ell+1$ in the second:

$$\begin{aligned} z_k &= \sum_{\ell=0}^{n/2-1} f_{2\ell} e^{-2\pi jk2\ell/n} + \sum_{\ell=0}^{n/2-1} f_{2\ell+1} e^{-2\pi jk(2\ell+1)/n} \\ &= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk2\ell/n} + \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{o}} e^{-2\pi jk(2\ell+1)/n} \end{aligned}$$

where

$$\vec{f}^{\mathsf{e}} = \begin{pmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \end{pmatrix} \in \mathbb{R}^{n/2}, \quad \vec{f}^{\mathsf{o}} = \begin{pmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \end{pmatrix} \in \mathbb{R}^{n/2}.$$

Applying the laws of the exponential function and $2/n = 1/(n/2)$ we obtain

$$\begin{aligned} z_k &= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk\ell/(n/2)} + \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{o}} e^{-2\pi jk\ell/(n/2)} e^{-2\pi jk/n} \\ &= \underbrace{\sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk\ell/(n/2)}}_{a_k} + \underbrace{\left(\sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{o}}\ e^{-2\pi jk\ell/(n/2)}\right)}_{b_k} e^{-2\pi jk/n} \\ &= a_k + b_k e^{-2\pi jk/n}. \end{aligned}$$

The factor $e^{-2\pi jk/n}$ could be factored out in the second sum because it is independent of ℓ.

The crucial point, which gives us a reduction in the number of multiplications is that

$$a_k = a_{k+n/2} \text{ and } b_k = b_{k+n/2}.$$

If one has to compute z_k for a certain k and has therefore computed a_k and b_k one gets without additional effort $z_{k+n/2}$ for free. This means that we get two Fourier Coefficients for the prize of one. This can be seen for a_k (and analogously for b_k) as follows.

$$\begin{aligned}
a_{k+n/2} &= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi j(k+n/2)\ell/(n/2)} \\
&= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk\ell/(n/2)} e^{-2\pi j(n/2)\ell/(n/2)} \\
&= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk\ell/(n/2)} \underbrace{e^{-2\pi j\ell}}_{=1} \\
&= a_k
\end{aligned}$$

Therefore it holds that

$$\begin{aligned}
z_{k+n/2} &= a_{k+n/2} + b_{k+n/2} e^{-2\pi j(k+n/2)/n} \\
&= a_k + b_k e^{-2\pi jk/n} e^{-2\pi j(n/2)/n} \\
&= a_k + b_k e^{-2\pi jk/n} \underbrace{e^{-\pi j}}_{=-1} \\
&= a_k - b_k e^{-2\pi jk/n}.
\end{aligned}$$

Therefore we have to compute for $k = 0, \dots, n/2 - 1$:

$$\begin{aligned}
a_k &= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{e}} e^{-2\pi jk\ell/(n/2)} \\
b_k &= \sum_{\ell=0}^{n/2-1} f_\ell^{\mathsf{o}} e^{-2\pi jk\ell/(n/2)} \\
z_k &= a_k + b_k e^{-2\pi jk/n} \\
z_{k+n/2} &= a_k - b_k e^{-2\pi jk/n}
\end{aligned}$$

Now we can assemble the a_k and b_k, which we have to compute only for $k = 0, \dots, n/2 - 1$ into vectors with $n/2$ components each:

$$\begin{aligned}
\vec{a} &= B_{n/2}^* \vec{f}^{\mathsf{e}} \\
\vec{b} &= B_{n/2}^* \vec{f}^{\mathsf{o}}.
\end{aligned}$$

These are the above mentioned two $n/2$-DFTs, whose evaluation costs $n^2/2$ multiplications. In addition we need $n/2$ multiplications for the products

$$b_k e^{-2\pi jk/n}, \qquad k = 0, \dots, n/2 - 1.$$

The total effort is therefore

$$\frac{n^2}{2} + \frac{n}{2} \quad \text{multiplications.}$$

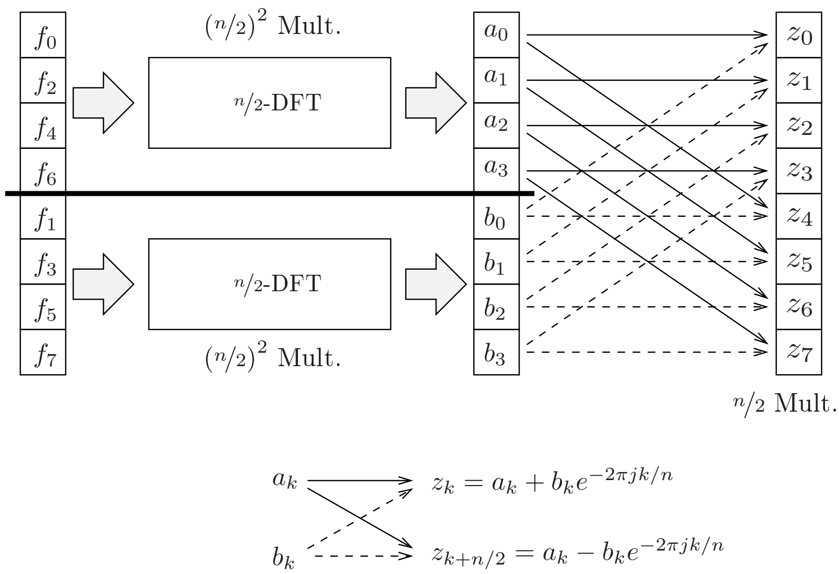


Figure 5.2: Reduction of a DFT of order $n = 8$ to two DFTS of order $n/2 = 4$.

Recursion. The same tricl is now applied to reduce the two DFTs of order $n/2$ to four DFTs of order $n/4$, see Figure 5.3. Four DFTs of order $n/4$ cost

$$4 \times \left(\frac{n}{4}\right)^2 = \frac{n^2}{4} \text{ multiplications.}$$

In addition there are $n/2$ multiplications to recombine the results of the DFTs of order $n/4$ and $n/2$ multiplications to recombine the results of the DFTs of order $n/2$. The total effort is therefore $n^2/4 + n$.

This process can be continued until it ends with DFTs or order $n = 1$, which is the case after $\mathrm{ld}(n)$ steps. As a DFT of order $n = 1$ costs nothing the only remaining costs are $n/2$ multiplications for the recombinations, i.e. in total

$$\frac{1}{2} n \mathrm{ld}(n) \text{ multiplications.}$$

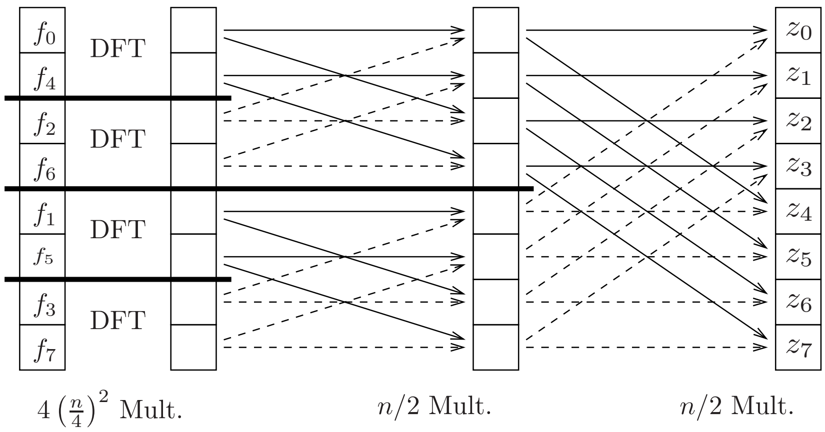


Figure 5.3: Reduction of a DFT of order $n = 8$ to four DFTs of order $n/4 = 2$.

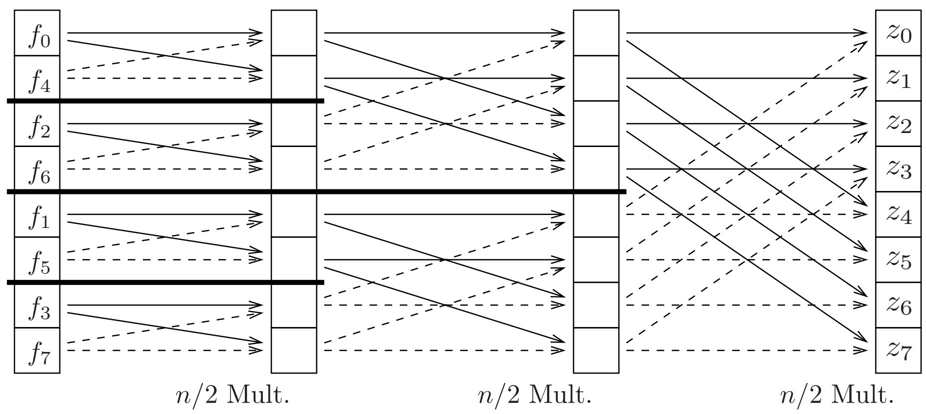


Figure 5.4: FFT of order $n = 8$ costs $\frac{1}{2}n\mathrm{ld}(n) = 12$ complex multiplications.

Bit Reverse. As we see in Figure 5.3, the coefficients of the input vector $\vec{f}$ are permuted first before they enter the DFT chain. For $n =$ the order is

$$f_0, f_4, f_2, f_6, f_1, f_5, f_3, f_7.$$

The underlying principle of this permutation becomes clear when the indices are written in binary notation and compared to the natural order:

0	4	2	6	1	5	3	7
000	100	010	110	001	101	011	111
000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

The order in which the $\vec{f}$-components enter the chain is obtained by presenting the numbers $0, \dots, n-1$ binary, reversing the bit order and converting back to decimal system. Correspondingly for $n = 16$ we obtain the following order:

0	1	2	3	4	5	...	14	15
0000	0001	0010	0011	0100	0101	...	1110	1111
0000	1000	0100	1100	0010	1010	...	0111	1111
0	8	4	12	2	10	...	7	15

6 Properties of the DFT

Let $\vec{f} \in \mathbb{R}^n$ be a vector whose components are the samples of a signal. The components of $\vec{f}$ are addressed with indices 0 to $n-1$. The DFT of $\vec{f}$ is denoted by $\vec{F}$, i.e.

$$\vec{F} \quad = \quad \mathrm{DFT}(\vec{f}).$$

According to the definition of the FFT on page 61 it holds that

$$\vec{F} \quad = \quad \frac{1}{n} B^* \vec{f}$$

or

$$F_k \quad = \quad \frac{1}{n} \sum_{\ell=0}^{n-1} f_\ell e^{-2\pi j k \ell / n}.$$

We will apply the same notation as for the Fourier Transform and write

$$\vec{f} \quad \circ\!\!-\!\!\bullet \quad \vec{F}$$

or

$$f_\ell \quad \circ\!\!-\!\!\bullet \quad F_k.$$

Modulo Function For every integer $\ell \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define $\ell \bmod n$ (read ℓ modulo n) as

$$\ell \bmod n \quad = \quad \ell + qn$$

where $q \in \mathbb{Z}$ is such that

$$0 \leq \ell + qn \quad < \quad n.$$

In order to obtain $\ell \bmod n$ we begin with ℓ and repeatedly add or subtract n until the result is between 0 and $n-1$. For example

$$\begin{aligned} 7 \bmod 3 &= 1 \\ 9 \bmod 3 &= 0 \\ -5 \bmod 3 &= 1 \\ -1 \bmod 3 &= 2 \\ 0 \bmod 3 &= 0. \end{aligned}$$

If ℓ is positive, then $\ell \bmod n$ is simply the rest of integer division ℓ by n.

Some properties of the modulo function:

$$\begin{aligned} \ell \bmod n &= \ell \qquad \text{for } \ell = 0, 1, \ldots, n-1 \\ n \bmod n &= 0 \\ -1 \bmod n &= n-1. \end{aligned}$$

6.1 Cyclic Shift

Let $\vec{g} \in \mathbb{R}^n$ be the vector obtained by cyclic shifting the components of $\vec{f} \in \mathbb{R}^n$ one place, i.e.

$$g_\ell = f_{\ell-1} \quad \text{for } \ell = 1, 2, \ldots, n-1$$

and

$$g_0 = f_{n-1}.$$

Using modulo function this can be written in one line:

$$g_\ell = f_{(\ell-1) \bmod n}, \quad \ell = 0, 1, \ldots, n-1.$$

As shown in Figure 6.1, cyclic shifting means that all samples of f are shifted one place to the right and the last sample moves to the beginning.

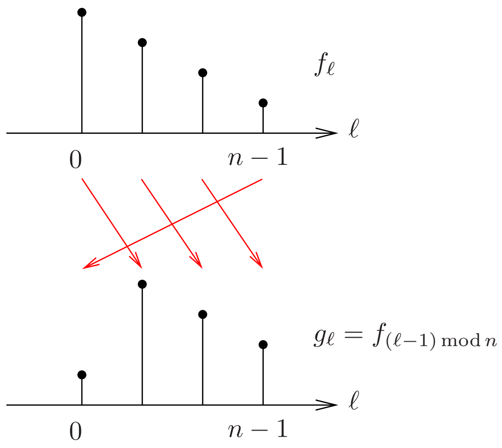


Figure 6.1: Cyclic Shift

Cyclic shift in the time domain has its correspondence in the frequency domain which is very similar to the Fourier Transform.

Theorem 6.1 (Cyclic Shift)

$$f_{(\ell-1) \bmod n} \quad \circ\!\!-\!\!\bullet \quad e^{-2\pi jk/n} F_k$$

The proof is as follows:

$$
\begin{aligned}
f_{(\ell-1) \bmod n} \quad &\circ\!\!-\!\!\bullet \quad \frac{1}{n}\sum_{\ell=0}^{n-1} f_{(\ell-1) \bmod n}\, e^{-2\pi jk\ell/n} \\
&= \quad e^{-2\pi jk/n}\frac{1}{n}\sum_{\ell=0}^{n-1} f_{(\ell-1) \bmod n}\, e^{-2\pi jk(\ell-1)/n} \\
&= \quad e^{-2\pi jk/n}\frac{1}{n}\sum_{\ell=-1}^{n-2} f_{\ell \bmod n}\, e^{-2\pi jk\ell/n} \\
&= \quad e^{-2\pi jk/n}\frac{1}{n}\left(\sum_{\ell=0}^{n-2} f_\ell\, e^{-2\pi jk\ell/n} + f_{n-1}e^{-2\pi jk(-1)/n}\right) \\
&= \quad e^{-2\pi jk/n}\frac{1}{n}\left(\sum_{\ell=0}^{n-2} f_\ell\, e^{-2\pi jk\ell/n} + f_{n-1}e^{-2\pi jk(n-1)/n}\right) \\
&= \quad e^{-2\pi jk/n}\frac{1}{n}\sum_{\ell=0}^{n-1} f_\ell\, e^{-2\pi jk\ell/n} \\
&= \quad e^{-2\pi jk/n}F_k
\end{aligned}
$$

As an immediate consequence we obtain the following correspondence for cyclic shift by m places:

$$
f_{(\ell-m) \bmod n} \quad \circ\!\!-\!\!\bullet \quad e^{-2\pi jkm/n}F_k
$$

6.2 Cyclic Convolution

As shown in Section 3.2 we can filter a signal by discrete convolution. However, this process is expensive. The convolution of two sequences of length n costs n^2 multiplications. The convolution theorem suggests that discrete convolution can be implemented more efficiently in the frequency domain by component wise multiplication, which costs only n multiplications. Yet, there is a complication in the discrete case. Multiplication in frequency domain means not convolution in time doman but a variation of it called *cyclic* convolution. However, we will rectify this later on by showing how convolution can be done using cyclic convolutions, which we can do efficiently via FFT.

Definition 6.2 (Cyclic Convolution)
The cycilc convolution $\otimes \in \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is defined by*

$$(\vec{f} \otimes \vec{g})_\ell = \sum_{m=0}^{n-1} g_m f_{(\ell-m) \bmod n}.$$

Cyclic convolution has the same properties as normal convolution, like commutativity and linearity:

$$\begin{aligned} \vec{f} \otimes \vec{g} &= \vec{g} \otimes \vec{f} \\ (c\vec{f}) \otimes \vec{g} &= c(\vec{f} \otimes \vec{g}) \\ (\vec{f}_1 + \vec{f}_2) \otimes \vec{g} &= (\vec{f}_1 \otimes \vec{g}) + (\vec{f}_2 \otimes \vec{g}). \end{aligned}$$

Important for us is the interrelation between cyclic convolution and DFT.

Theorem 6.3 (Cyclic Convolution Theorem)
Cyclic convolution in time domain corresponds to component wise multiplication in frequency domain, i.e.

$$\frac{1}{n}(\vec{f} \otimes \vec{g})_\ell \quad \circ\!\!-\!\!\bullet \quad F_k G_k.$$

Theorem 6.3 can be rewritten in vector notation as

$$\frac{1}{n}(\vec{f} \otimes \vec{g}) \quad \circ\!\!-\!\!\bullet \quad \vec{F}\vec{G}$$

where the multiplication of the vectors on the right hand side is done component wise.

Computational Costs. Theorem 6.3 is important because it can be used to reduce the costs for cyclic convolution.

- The computation of

$$(\vec{f} \otimes \vec{g})_\ell = \sum_{m=0}^{n-1} g_m f_{(\ell-m) \bmod n}$$

costs n multiplications. The computation of

$$\vec{f} \otimes \vec{g}$$

costs therefore n^2 multiplications.

- We can do cyclic convolution much faster via FFT. Multiplying both sides of the correspondence with n^2 we obtain

$$n(\vec{f} \otimes \vec{g}) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad (n\vec{F})(n\vec{G}).$$

The computation of $n\vec{F}$ and $n\vec{G}$ by FFT costs

$$n\mathrm{ld}(n) \text{ multiplications.}$$

Component wise multiplication $\vec{H} = \vec{F}\vec{G}$ costs n multiplications. We obtain

$$n(\vec{f} \otimes \vec{g}) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad n^2\vec{H}$$

or

$$\frac{1}{n}(\vec{f} \otimes \vec{g}) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \vec{H}.$$

Inverse FFT of $\vec{H}$ costs

$$\frac{1}{2}n\mathrm{ld}(n) \text{ multiplications.}$$

Finally each component of the result has to be multiplied by n in order to obtain $\vec{f} \otimes \vec{g}$, which costs another n multiplications. The total effort is therefore

$$n\mathrm{ld}(n) + n + \frac{1}{2}n\mathrm{ld}(n) + n \quad = \quad n\left(\frac{3}{2}\mathrm{ld}(n) + 2\right).$$

For $n = 256$ cyclic convolution in time domain costs 65 536 multiplications, the indirect route via FFT only 3 584.

Now for the proof of Theorem 6.3:

$$
\begin{aligned}
F_k G_k &= F_k \underbrace{\frac{1}{n}\sum_{m=0}^{n-1} g_m\, e^{-2\pi jkm/n}}_{G_k} \\
&= \frac{1}{n}\sum_{m=0}^{n-1} F_k g_m\, e^{-2\pi jkm/n} \\
&= \frac{1}{n}\sum_{m=0}^{n-1} \underbrace{\frac{1}{n}\sum_{\ell=0}^{n-1} f_\ell\, e^{-2\pi jk\ell/n}}_{F_k} g_m\, e^{-2\pi jkm/n} \\
&= \frac{1}{n^2}\sum_{m=0}^{n-1}\sum_{\ell=0}^{n-1} g_m f_\ell\, e^{-2\pi jk(\ell+m)/n} \\
&= \frac{1}{n^2}\sum_{m=0}^{n-1}\sum_{\ell=m}^{m+n-1} g_m f_{\ell-m}\, e^{-2\pi jk\ell/n} \\
&= \frac{1}{n^2}\sum_{m=0}^{n-1}\left(\sum_{\ell=m}^{n-1} g_m f_{\ell-m}\, e^{-2\pi jk\ell/n} + \sum_{\ell=n}^{m+n-1} g_m f_{\ell-m}\, e^{-2\pi jk\ell/n}\right) \\
&= \frac{1}{n^2}\sum_{m=0}^{n-1}\left(\sum_{\ell=m}^{n-1} g_m f_{\ell-m}\, e^{-2\pi jk\ell/n} + \sum_{\ell=0}^{m-1} g_m f_{(\ell-m) \bmod n}\, e^{-2\pi jk(\ell+n)/n}\right) \\
&= \frac{1}{n^2}\sum_{m=0}^{n-1}\sum_{\ell=0}^{n-1} g_m f_{(\ell-m) \bmod n}\, e^{-2\pi jk\ell/n} \\
&= \frac{1}{n^2}\sum_{\ell=0}^{n-1}\underbrace{\sum_{m=0}^{n-1} g_m f_{(\ell-m) \bmod n}}_{(\vec{f}\otimes\vec{g})_\ell}\, e^{-2\pi jk\ell/n} \\
&= \frac{1}{n^2}\sum_{\ell=0}^{n-1} (\vec{f}\otimes\vec{g})_\ell\, e^{-2\pi jk\ell/n} \\
&\bullet\!\!-\!\!\circ\ \frac{1}{n}(\vec{f}\otimes\vec{g})_\ell.
\end{aligned}
$$

Example 6.4 In analogy to the time continuous case there is a discrete Dirac impulse defined by

$$\vec{\delta} \in \mathbb{R}^n, \qquad \delta_\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{else.} \end{cases}$$

As expected it holds that

$$\vec{f} \quad = \quad \vec{f} \otimes \vec{\delta}.$$

From the definition of the cyclic convolution it follows that

$$(\vec{f} \otimes \vec{\delta})_\ell \quad = \quad \sum_{m=0}^{n-1} \delta_m f_{(\ell-m) \bmod n}.$$

In this sum all summands are zero except the one for $m = 0$. Hence

$$\sum_{m=0}^{n-1} \delta_m h_{(\ell-m) \bmod n} \quad = \quad f_\ell.$$

The same result is obtained using DFT. The DFT of the Dirac impulse is

$$\begin{aligned} \delta_\ell \quad \circ\!\!-\!\!\bullet \quad & \frac{1}{n} \sum_{\ell=0}^{n-1} \delta_\ell e^{-2\pi jk\ell/n} \\ = \quad & \frac{1}{n} e^{-2\pi jk0/n} \\ = \quad & 1/n. \end{aligned}$$

Using Theorem 6.3 it holds that

$$\frac{1}{n}(\vec{f} \otimes \vec{\delta})_\ell \quad \circ\!\!-\!\!\bullet \quad F_k \frac{1}{n}$$

and therefore

$$(\vec{f} \otimes \vec{\delta})_\ell \quad \circ\!\!-\!\!\bullet \quad F_k.$$

As

$$f_\ell \quad \circ\!\!-\!\!\bullet \quad F_k$$

it follows that

$$(\vec{f} \otimes \vec{\delta})_\ell \quad = \quad f_\ell.$$

Example 6.5 Let g_ℓ be the Dirac impulse shifted by one, i.e.

$$\vec{g} \in \mathbb{R}^n, \qquad g_\ell = \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{else.} \end{cases}$$

For the cyclic convolution of $\vec{f}$ and $\vec{g}$ it holds that

$$(\vec{f} \otimes \vec{g})_\ell = \sum_{m=0}^{n-1} g_m f_{(\ell-m) \bmod n}.$$

In this sum every summand is zero except the one for $m = 1$. Hence

$$\sum_{m=0}^{n-1} g_m f_{(\ell-m) \bmod n} = f_{(\ell-1) \bmod n}.$$

It follows that

$$(\vec{f} \otimes \vec{g})_\ell = f_{(\ell-1) \bmod n},$$

i.e. the effect of cyclic convolution of $\vec{f}$ with $\vec{g}$ is cyclic shift by one place. The same result is obtained using Theorem 6.3. The DFT of $\vec{g}$ is

$$\begin{aligned} G_k &= \frac{1}{n} \sum_{\ell=0}^{n-1} g_\ell e^{-2\pi jk\ell/n} \\ &= \frac{1}{n} e^{-2\pi jk/n}. \end{aligned}$$

Hence

$$\frac{1}{n} (\vec{f} \otimes \vec{g})_\ell \quad \circ\!\!-\!\!\bullet \quad F_k \frac{1}{n} e^{-2\pi jk/n}$$

or

$$(\vec{f} \otimes \vec{g})_\ell \quad \circ\!\!-\!\!\bullet \quad e^{-2\pi jk/n} F_k.$$

According to Theorem 6.1 it holds that

$$f_{(\ell-1) \bmod n} \quad \circ\!\!-\!\!\bullet \quad e^{-2\pi jk/n} F_k$$

and therefore

$$(\vec{f} \otimes \vec{g})_\ell = f_{(\ell-1) \bmod n}.$$

7 Fast Convolution with FFT

In the previous section we saw that cyclic convolution can be done very efficiently with FFT. Next we will show how linear[3] convolution can be implemented using cyclic convolutions. In this way linear convolution, which plays a key role in digital signal processing, can be done efficiently with FFTs.

Linear convolution of two signals f and g is defined by

$$(f * g)_\ell \quad = \quad \sum_{m=-\infty}^{\infty} g_m f_{\ell-m}.$$

In many applications g is the impulse response of an FIR filter and therefore non-zero only in a finite interval $\ell = 0, \ldots, G-1$.

$$g_\ell \quad = \quad 0 \text{ for } \ell \notin [0, G-1].$$

In this case convolution simplifies to a finite sum

$$h_\ell \; = \; (f * g)_\ell \; = \; \sum_{m=0}^{G-1} g_m f_{\ell-m}.$$

The computation of h_ℓ can be realized by an inner product of two vectors with G components each.

Convolution of a Signal Block. We make the preliminary assumption that not only g but also f is non-zero only in some interval $\ell = 0, \ldots, F-1$ with $F \geq G$, i.e.

$$f_\ell = 0 \text{ for } \ell \notin [0, F-1].$$

Figure **??** illustrates this for $G = 4$ and $F = 9$. The dashed lines indicate that the signals are zero in this region.

Let $\vec{g}, \vec{f} \in \mathbb{R}^F$ be vectors whose components are given by the first F samples of g and f. Linear convolution $f * g$ is obviously not identical with cyclic convolution $\vec{f} \otimes \vec{g}$. However, the result is the same for certain values of ℓ. From the example in Figure 7.2 we see that linear and cyclic convolution conicide for $\ell = 3, \ldots, 8$ because there is no "wrapping" in cyclic convolution. In general it holds that

$$(\vec{f} \otimes \vec{g})_\ell \quad = \quad (f * g)_\ell \text{ for } \ell = G-1, \ldots, F-1.$$

This should be obvious from a comparision of Figure 7.1 and 7.2 but can be

[3] In order to distinguish "normal" convolution from cyclic convolution we will use the term linear convolution for the former.

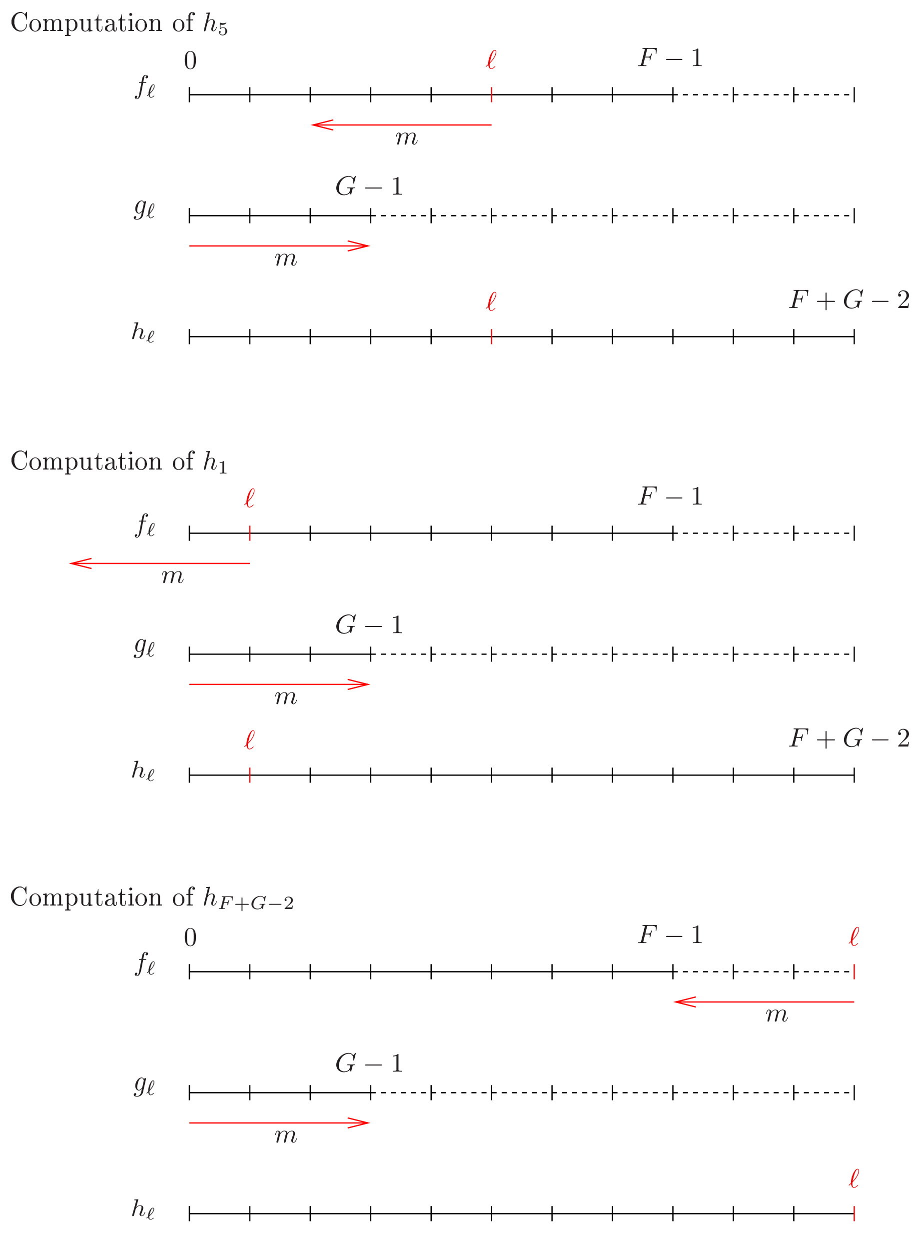


Figure 7.1: Linear convolution $h_\ell = (f * g)_\ell$ for $G = 4$ and $F = 9$

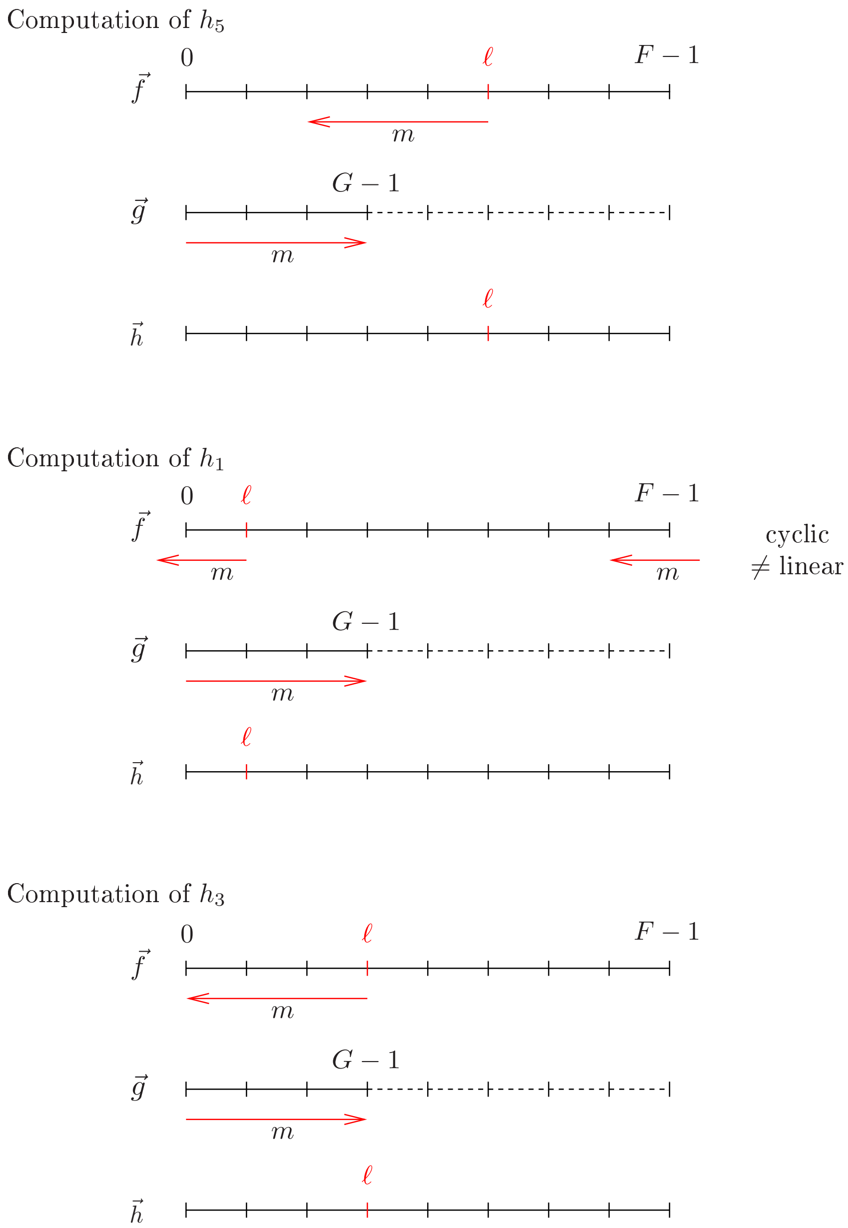


Figure 7.2: Cyclic convolution $\vec{h} = \vec{f} \otimes \vec{g}$ for $G = 4$ and $F = 9$.

verified as follows. Let $G-1 \le \ell \le F-1$. Then it holds that

$$\begin{aligned}(\vec{f} \otimes \vec{g})_\ell &= \sum_{m=0}^{F-1} g_m f_{(\ell-m) \bmod F} \\ &= \sum_{m=0}^{G-1} g_m f_{(\ell-m) \bmod F} \qquad \text{da } g_m = 0 \text{ f"ur } m \ge G \\ &= \sum_{m=0}^{G-1} g_m f_{\ell-m} \qquad \text{da } \ell \ge G-1 \\ &= (f * g)_\ell.\end{aligned}$$

Convolution of an unlimited Signal. If f is not limited in time or a signal with long duration we cut out successively blocks of length F from f and convolve them cyclically with the impulse response g. The first $G-1$ values of the results are useless as seen above, the remaining $F-G+1$ values coincide with linear convolution. Thus, if the sample blocks from f overlap by $G-1$ values we can compute that way the entire signal $f * g$. In each cyclic convolution we obtain $F-G+1$ values of the signal $h = f * g$ and despose the remaining first $G-1$ values, see Figure 7.3.

The precise calculation of the indices is a bit tedious but has to be done for the implementation in a computer program. The first block contains the samples

$$f_{-G+1} \text{ to } f_{F-G}$$

and gives after cyclic convolution with g and the elimination of the first $G-1$ values

$$h_0 \text{ bis } h_{F-G}.$$

In each of the subsequent block operations the indices shift by

$$\Delta = F - G + 1.$$

For $i = 0, 1, 2, \ldots$ the i-th block of f contains F samples

$$f_{-G+1+i\Delta} \text{ to } f_{F-G+i\Delta}.$$

From the result of cyclic convolution of this block with g the first $G-1$ values are disposed. The remaining $F-G+1$ values are

$$h_{i\Delta} \text{ to } h_{F-G+i\Delta}.$$

The algorithm in pseudo code looks as follows:

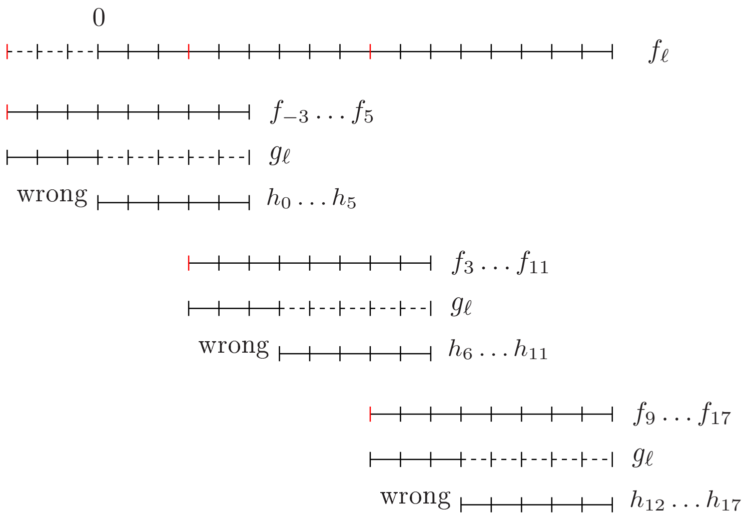


Figure 7.3: Convolution $h = f * g$ for $G = 4$ and $F = 9$.

Program fast convolution with FFT

Input:
Impulse response $g_0, \dots, g_G - 1$
Input signal $f_0, f_1, \dots$

Output:
Outputsignal $h_0, h_1, \dots$

$\vec{g} = (g_0, g_1, \dots, g_{G-1}, 0, 0, \dots, 0)^{\mathrm{T}}$	*Impulse response as F-dimensional vector*
$h = \langle\rangle$	*Initialise h als empty sequence*
$\Delta = F - G + 1$	*Shift width*
$s = -G + 1$	*Start index*
while(true)	
$\vec{f} = (f_s, f_{s+1}, f_{s+F-1})^{\mathrm{T}}$	*F-dimensional sample block*
$\vec{z} = \vec{f} \otimes \vec{g}$	*Cyclic convolution with FFT*
$h = \mathrm{concat}(h, z_{G-1}, \dots, z_{F-1})$	*Take correct sample values*
$s = s + \Delta$	*Shift start index*

Computational Cost of Fast Convolution. The number of multipications to compute $F-G+1$ correct samples in one iteration of the the above algorithm can be estimated as follows. Let $\vec{F}$ and $\vec{G}$ be the DFT of a block of f and the zero padded impulse response. The computation of

$$\frac{1}{n}\vec{G}$$

can be done in advance. The cyclic convolution

$$\vec{z} \quad = \quad \vec{f} \otimes \vec{g}$$

consists of the following steps:

- The FFT of $\vec{f}$ costs

  $$\frac{1}{2}F\mathrm{ld}(F)$$

  multiplications.
- Computation of

  $$\frac{1}{n}\vec{F}\vec{G}$$

  costs F multiplications.
- Its inverse FFT costs another

  $$\frac{1}{2}F\mathrm{ld}(F)$$

  multiplications.

The effort *per sample* of fast convolution is therefore

$$\frac{F\mathrm{ld}(F)+F}{F-G+1}.$$

In order to obtain the optimal block length F for a given filter length G we have to find the zero of the derivative of this term with respect to F. This leads to a nonlinear equation which cannot be solved in closed form. However, as F has to be a power of two, the optimal block length can easily be determined.

Example 7.1 With a filter length $G = 100$ we obtain an optimal block length $F = 1024$ and about 12.18 multiplicaion per sample with fast convolution. This is less than 100 multiplications per sample with straight forward convolution in time domain. However, we should take into account that fast convolution uses complex multiplications whereas the multiplications in time domain are real.

Example 7.2 With a filter length $G = 1024$ we obtain an optimal block length $F = 8192$ and about 16 multiplications per sample with fast convolution, which is much less than 1024 multiplications per sample with convolution in time domain. Even if we respect that a complex multiplication costs 4 real multiplications, we still obtain a speedup of factor 16.

8 Amplitude Modulation

Let $f(t)$ be a band limited signal with cutoff frequency ω_c, i.e.

$$F(\omega) \quad = \quad 0 \quad \text{for } |\omega| \geq \omega_c.$$

Further let $\hat{\omega} \geq \omega_c$. According to Theorem 1.15 it holds that

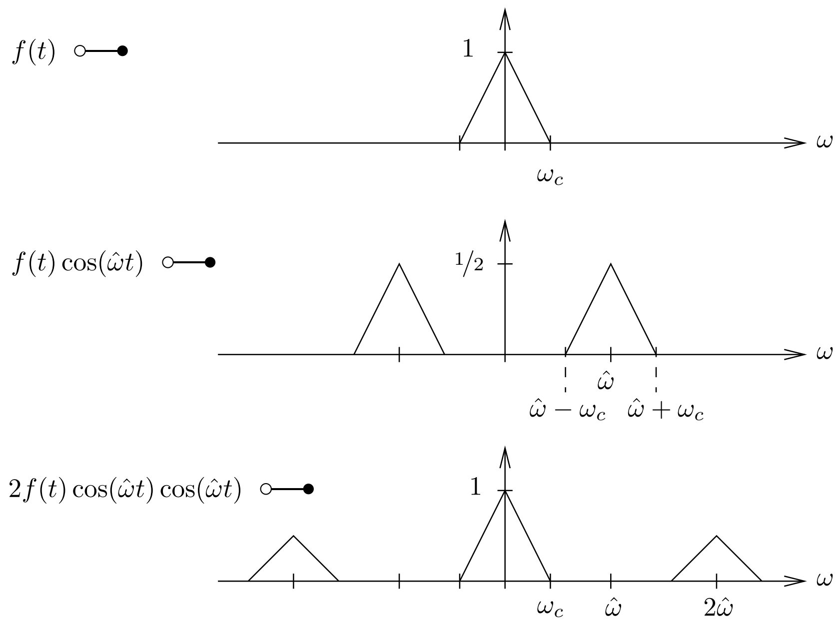
$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega-\hat{\omega}) + F(\omega+\hat{\omega})).$$

We obtain two copies of $F(\omega)$ shifted by $\pm\hat{\omega}$. As $\omega_c \geq \hat{\omega}$ the two copies do not overlap. While the original signal $f(t)$ coverd the frequency band $[0, \omega_c]$, the modulated signal $f(t)\cos(\hat{\omega}t)$ covers $[\hat{\omega}-\omega_c, \hat{\omega}+\omega_c]$.

Now let us multiply $f(t)\cos(\hat{\omega}t)$ once more by $\cos(\hat{\omega}t)$ and scale by factor 2. We obtain

$$\begin{aligned} 2f(t)\cos(\hat{\omega}t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad & \frac{2}{4}(F(\omega-2\hat{\omega}) + F(\omega) + F(\omega) + F(\omega+2\hat{\omega})) \\ = \quad & F(\omega) + \frac{1}{2}(F(\omega-2\hat{\omega}) + F(\omega+2\hat{\omega})). \end{aligned}$$

The result consists of the original signal $F(\omega)$ and two copies shifted by $\pm 2\hat{\omega}$. In order to recover $F(\omega)$ from this signal we apply a low pass filter with cutoff frequency $\hat{\omega}$, which deletes the shifted copies.



This procedure is called amplitude modulation and is used to transmit two band limited signals f_1, f_2 with cutoff frequency ω_c simultaneously over a common medium like a cable or air in case of wireless communication. All we have to do is choose suitable modluation frequencies $\hat{\omega}_1$, $\hat{\omega}_2$ such that the copies do not overlap in frequency domain.

$$\begin{aligned} \hat{\omega}_{1,2} &\geq \omega_c \\ |\hat{\omega}_2 - \hat{\omega}_1| &\geq 2\omega_c. \end{aligned}$$

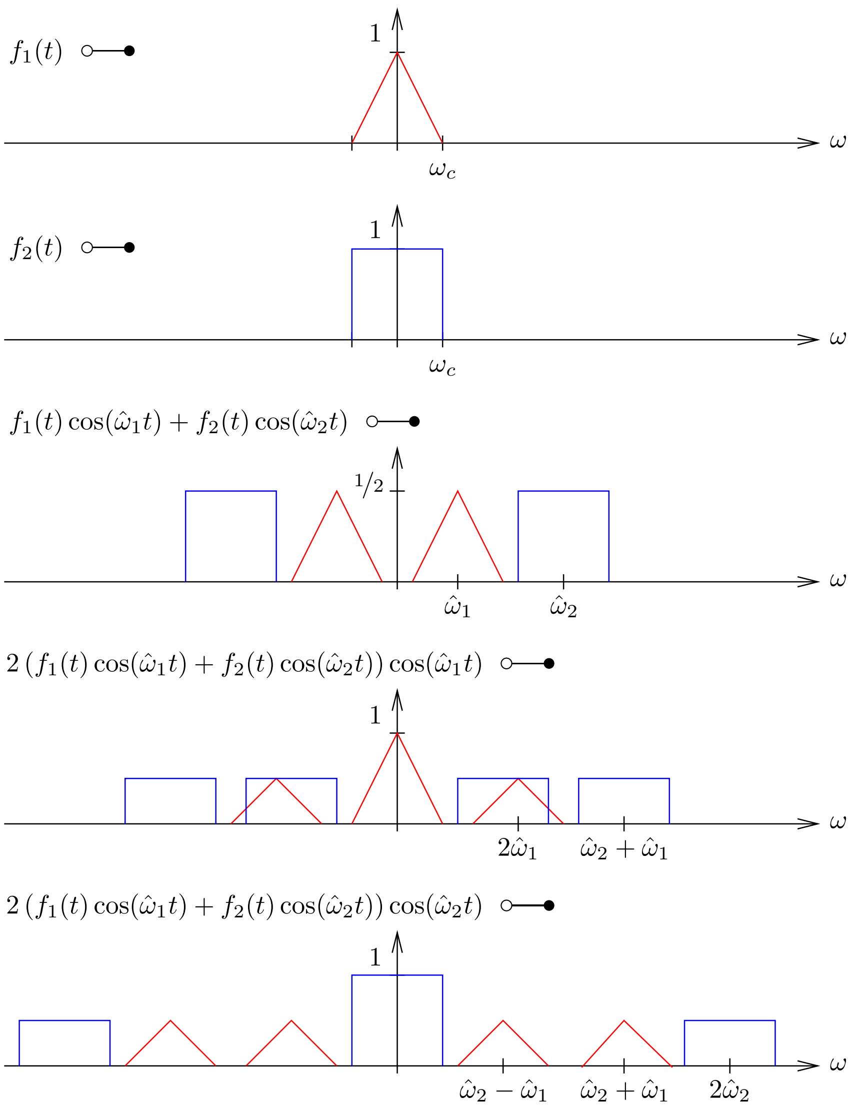
The transmitted signal is

$$f_1(t)\cos(\hat{\omega}_1 t) + f_2(t)\cos(\hat{\omega}_2 t).$$

If the receiver wants to recover $f_i(t)$ he demodulates with $2\cos(\hat{\omega}_i t)$ and obtains

$$2(f_1(t)\cos(\hat{\omega}_1 t) + f_2(t)\cos(\hat{\omega}_2 t))\cos(\hat{\omega}_i t), \qquad i = 1, 2.$$

Low pass filtering with cutoff frequency ω_c gives $f_i(t)$.



8.1 Quadrature Amplitude Modulation

Quadrature amplitude modulation is a technique to use bandwidth more efficiently. It allows us to transmit two signals $f(t)$ and $g(t)$ simultaneously in the same frequency band, i.e. using the same modulation frequency $\hat{\omega}$. All we have to do is modulate $f(t)$ with $\cos(\hat{\omega}t)$ and $g(t)$ with $\sin(\hat{\omega}t)$.

If the receiver demodulates with $2\cos(\hat{\omega}t)$ he obtains $f(t)$ after low pass filtering. Demodulating with $2\sin(\hat{\omega}t)$ and low pass filtering gives $g(t)$.

This can be seen as follows. Let f, g be band limited signals with cutoff frequency ω_c, i.e.

$$F(\omega) \;=\; G(\omega) \;=\; 0 \quad \text{for } |\omega| \geq \omega_c.$$

Assume the modulation frequency $\hat{\omega}$ is large enough, i.e. $\hat{\omega} \geq \omega_c$.

The transmitted signal is

$$f(t)\cos(\hat{\omega}t) + g(t)\sin(\hat{\omega}t).$$

If the receiver demodulates with $2\cos(\hat{\omega}t)$, he obtains

$$\underbrace{2f(t)\cos(\hat{\omega}t)\cos(\hat{\omega}t)}_{\circ\!-\!\bullet A(\omega)} + \underbrace{2g(t)\sin(\hat{\omega}t)\cos(\hat{\omega}t)}_{\circ\!-\!\bullet B(\omega)}.$$

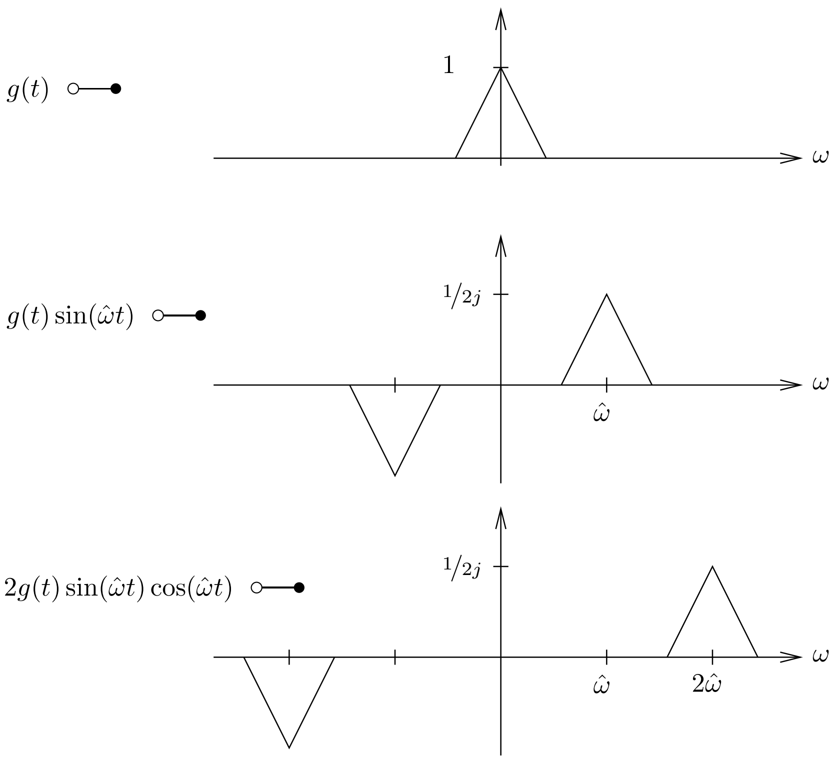
From the previous section we know that

$$A(\omega) \;=\; F(\omega) \quad \text{for } |\omega| \leq \omega_c.$$

So it remains to show that

$$B(\omega) \;=\; 0 \quad \text{for } |\omega| \leq \omega_c.$$

This can be easily derived from the following picture.



The proof follows from Theorem 1.15.

$$\begin{aligned}
g(t)\sin(\hat\omega t) \quad &\circ\!\!-\!\!\bullet \quad \frac{1}{2j}(G(\omega-\hat\omega) - G(\omega+\hat\omega)) \\
2g(t)\sin(\hat\omega t)\cos(\hat\omega t) \quad &\circ\!\!-\!\!\bullet \quad \frac{1}{2j}(G(\omega-2\hat\omega) - G(\omega) + G(\omega) - G(\omega+2\hat\omega)) \\
&= \quad \frac{1}{2j}(G(\omega-2\hat\omega) - G(\omega+2\hat\omega)) \quad = \quad B(\omega).
\end{aligned}$$

As $\hat\omega \geq \omega_c$ and $G(\omega) = 0$ for $|\omega| \geq \omega_c$, it follows that $B(\omega) = 0$ for $|\omega| \leq \omega_c$. The situation when the receiver demodulates with $2\sin(\hat\omega t)$ is analog.

8.2 Hilbert Transform

The Hilbert transform and the analytic signal are theoretical foundations for single side band amplitude modulation which needs only half the bandwidth compared to normal amplitude modulation.

Definition 8.1 (Hilbert Transform)
The Hilbert transform $\hat{f}$ of a signal f is defined as

$$\hat{f}(t) \quad = \quad \frac{1}{\pi t} * f(t).$$

Hilbert transform means simply convolution with $1/\pi t$. As

$$\frac{1}{\pi t} \quad \circ\!\!-\!\!\bullet \quad -j\text{sign}(\omega)$$

we obtain

$$\hat{F}(\omega) \quad = \quad -j\text{sign}(\omega)F(\omega).$$

This means that $F(\omega)$ is multiplied by a constant factor $-j$ for $\omega > 0$. The magnitude of $F(\omega)$ remains the same but $\pi/2$ is subtracted from the angle. All oscilations in $f(t)$ are phase shifted by $-\pi/2$.

For example, if $f(t) = \sin(\omega t)$ then $\hat{f}(t) = \sin(\omega t - \pi/2) = -\cos(\omega t)$.

If we apply the Hilbert transform twice on $f(t)$ we obtain in frequency domain

$$(-j\text{sign}(\omega))(-j\text{sign}(\omega))F(\omega) \quad = \quad -F(\omega), \qquad \omega \neq 0.$$

In time domain this means

$$\begin{aligned} \frac{1}{\pi t} * \hat{f}(t) \quad &= \quad -f(t) \\ f(t) \quad &= \quad -\frac{1}{\pi t} * \hat{f}(t). \end{aligned}$$

We can reconstruct the original signal $f(t)$ from its Hilbert transform $\hat{f}(t)$ simply by convolution with $-1/\pi t$.

Theorem 8.2 (Inverse Hilbert Transform)
Let $\hat{f}$ be the Hilbert transform of f. Then

$$f(t) \quad = \quad -\frac{1}{\pi t} * \hat{f}(t).$$

Example 8.3 The Hilbert transform of $\sin(t)$ is

$$\begin{aligned} \sin(t) * \frac{1}{\pi t} \quad \circ\!\!-\!\!\bullet \quad & -j\pi(\delta(\omega-1) - \delta(\omega+1))(-j\text{sign}(\omega)) \\ = \quad & -\pi(\delta(\omega-1) - \delta(\omega+1))\text{sign}(\omega) \\ = \quad & -\pi(\delta(\omega-1) + \delta(\omega+1)) \\ \bullet\!\!-\!\!\circ \quad & -\cos(t). \end{aligned}$$

From the above theorem follows that the inverse Hilbert transform of $\sin(t)$ is $\cos(t)$ and therefore the Hilbert transform of $\cos(t)$ is $\sin(t)$. This confirms that the Hilbert transform performs a phase shift by $\pi/2$.

Example 8.4 From the Fourier transforms

$$\begin{aligned} \frac{1}{t^2+1} &\quad \circ\!\!-\!\!\bullet \quad \pi e^{-|\omega|} \\ \frac{t}{t^2+1} &\quad \circ\!\!-\!\!\bullet \quad -j\pi \operatorname{sign}(\omega) e^{-|\omega|} \end{aligned}$$

we can directly see that the Hilbert transform of $\frac{1}{t^2+1}$ is $\frac{t}{t^2+1}$. and vice versa the Hilbert transform of $\frac{t}{t^2+1}$ is $-\frac{1}{t^2+1}$.

Example 8.5 The Hilbert transform of $1/\pi t$ is $-\delta(t)$ as

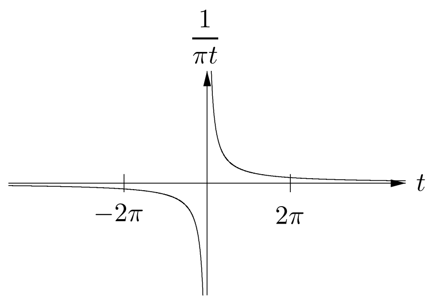
$$\frac{1}{\pi t} * \frac{1}{\pi t} \quad \circ\!\!-\!\!\bullet \quad (-j\operatorname{sign}(\omega))^2 \quad = \quad -1 \quad \bullet\!\!-\!\!\circ \quad -\delta(t).$$

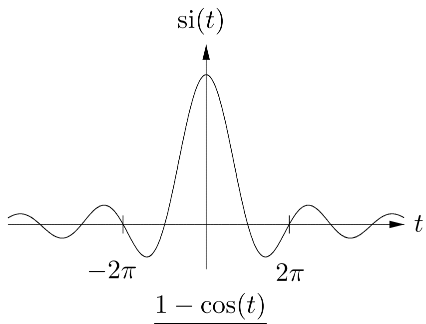
The si function plays an important role in signal processing because it is the impulse response of the ideal low pass filter, which is useful for signal reconstruction or D/A conversion. Fortunately it has a simple Hilbert transform.

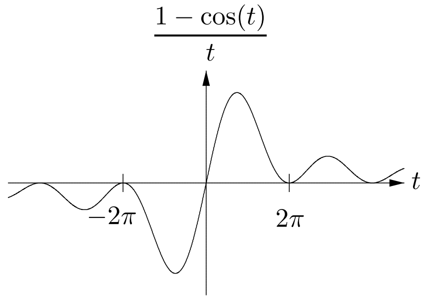
Theorem 8.6
Let $a > 0$. The Hilbert transform of $\mathrm{si}(at)$ *is*

$$\mathrm{si}(at) * \frac{1}{\pi t} \quad = \quad \frac{1-\cos(at)}{at}$$

for $t \neq 0$ and zero for $t = 0$.







With some good will we can see the $\pi/2$ phase shift between $\mathrm{si}(t)$ and $\frac{1-\cos(t)}{t}$ caused by the Hilbert transform. Note that the Hilbert transform of the si function can also be interpreted as the result of low pass filtering $1/\pi t$, which is equally visible in the above picture.

Proof. The Fourier transform of the si-function is

$$\mathrm{si}(t) \quad \circ\!\!-\!\!\bullet \quad \pi \mathrm{rect}_{-1,1}(\omega)$$

with

$$\mathrm{rect}_{a,b}(\omega) \quad = \quad \begin{cases} 1 & \text{if } a \le \omega \le b \\ 0 & \text{else} \end{cases}$$

Applying Theorem 1.16 we obtain

$$\mathrm{si}(at) \quad \circ\!\!-\!\!\bullet \quad \frac{\pi}{a}\mathrm{rect}_{-a,a}(\omega).$$

With the help of the Convolution Theorem we obtain

$$\begin{aligned}
\frac{1}{\pi t} * \mathrm{si}(at) \quad &\circ\!\!-\!\!\bullet \quad (-j\mathrm{sign}(\omega))\frac{\pi}{a}\mathrm{rect}_{-a,a}(\omega) \\
&= \begin{cases} -j\pi/a & \text{if } 0 < \omega \le a \\ j\pi/a & \text{if } 0 > \omega \ge -a \\ 0 & \text{else} \end{cases} \\
&\bullet\!\!-\!\!\circ \quad \frac{1}{2\pi}\left(\int_{-a}^{0} \frac{j\pi}{a} e^{j\omega t} d\omega - \int_{0}^{a} \frac{j\pi}{a} e^{j\omega t} d\omega\right) \\
&= \frac{j}{2a}\left(\int_{-a}^{0} e^{j\omega t} d\omega - \int_{0}^{a} e^{j\omega t} d\omega\right) \\
&= \frac{1}{2at}\left(\left[e^{j\omega t}\right]_{-a}^{0} - \left[e^{j\omega t}\right]_{0}^{a}\right) \qquad t \neq 0 \\
&= \frac{1}{2at}\left(1 - e^{-jat} - e^{jat} + 1\right) \\
&= \frac{1}{2at}\left(2 - (e^{jat} + e^{-jat})\right) \\
&= \frac{1}{2at}(2 - 2\cos(at)) \\
&= \frac{1 - \cos(at)}{at}.
\end{aligned}$$

The special case $t = 0$ is easy to see.

Theorem 8.7
If f is band limited with cutoff frequency ω_c then its Hilbert transform $\hat{f}$ is also band limited with cutoff frequency ω_c.

Proof. Follows immediately from

$$\hat{F}(\omega) \quad = \quad -j\mathrm{sign}(\omega)F(\omega)$$

We will now show how the Hilber transform can be computed with digital signal processing when signals are given as sequences of samples. We assume that $f(t)$ is band limited and the sampling interval T is large enough to avoid aliasing. We obtain the samples

$$f_k = f(kT), \qquad k \in \mathbb{Z}.$$

As shown in Section 3.1 we can reconstruct the analog signal f from its samples with the ideal low pass filter

$$f(t) = f_p(t) * \mathrm{si}(\pi t/T)$$

where

$$f_p(t) = \sum_k f_k \delta(t - kT)$$

is the pulse train of samples.

The Hilbert transform of f is

$$\begin{aligned} \hat{f}(t) &= f(t) * \frac{1}{\pi t} \\ &= f_p(t) * \mathrm{si}(\pi t/T) * \frac{1}{\pi t}. \end{aligned}$$

As convolution is associative, we can begin with the second product, which means low pass filtering of $1/\pi t$ or Hilbert transform of $\mathrm{si}(\pi t/T)$. This has already been done in Theorem 8.6 and with $a = \pi/T$ we obtain

$$\mathrm{si}\left(\frac{\pi t}{T}\right) * \frac{1}{\pi t} = \frac{T}{\pi t}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right)$$

for $t \neq 0$ and zero for $t = 0$.

Now using the properties of the Dirac pulse as in Section 3.2 we obtain

$$\begin{aligned} \hat{f}(t) &= f_p(t) * \frac{T}{\pi t}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right) \\ &= \sum_k f_k \delta(t - kT) * \frac{T}{\pi t}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right) \\ &= \sum_k f_k \frac{T}{\pi(t - kT)}\left(1 - \cos\left(\frac{\pi(t - kT)}{T}\right)\right). \end{aligned}$$

In order to be precise we should mention the exception for the k-th summand, which is zero in the special case $t = kT$.

As we are only interested in the samples of $\hat{f}$ we obtain

$$\begin{aligned}
\hat{f}_\ell &= \hat{f}(\ell T) \\
&= \sum_k f_k \frac{T}{\pi(\ell T - kT)}\left(1 - \cos\left(\frac{\pi(\ell T - kT)}{T}\right)\right) \\
&= \sum_k f_k \frac{1}{\pi(\ell - k)}(1 - \cos(\pi(\ell - k))) \\
&= \sum_k f_k \underbrace{\frac{1}{\pi(\ell - k)}(1 - \cos(\pi(\ell - k)))}_{g_{\ell-k}} \\
&= \sum_k f_k g_{\ell-k} \\
&= (f * g)_\ell
\end{aligned}$$

where the $*$ in the last line means discrete convolution. The (discrete) impulse response of the Hilbert transform is therefore

$$\begin{aligned}
g_k &= \begin{cases} \frac{1}{\pi k}(1 - \cos(\pi k)) & k \neq 0 \\ 0 & k = 0 \end{cases} \\
&= \begin{cases} \frac{2}{\pi k} & \text{if } k \text{ is odd} \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Hilbert transform of an analog signal f corresponds to convolving its samples f_k with g_k. Unfortunately the impulse response g_k is neither causal nor limited in time. However, for large $|k|$ the values of g_k are small and can be truncated without too much loss of accuracy. If we accept some time delay, the Hilbert transform of a signal can be computed in real time. The same optimizations with windowing as in Section 3.2 are applicable.

8.3 Analytic Signal

Let $f \in \mathbb{R} \to \mathbb{R}$ be a real signal with Fourier transform $F(\omega)$. As f is real, it holds that

$$F(-\omega) \quad = \quad \overline{F(\omega)}.$$

The values of $F(\omega)$ for negative ω are therefore redundant. This means that in signal transmission applications we can save half of the bandwidth!

The analytic signal f^+ of f is obtained by setting $F(\omega)$ to zero for negative ω. More precisely we define the analytic signal f^+ of f as follows.

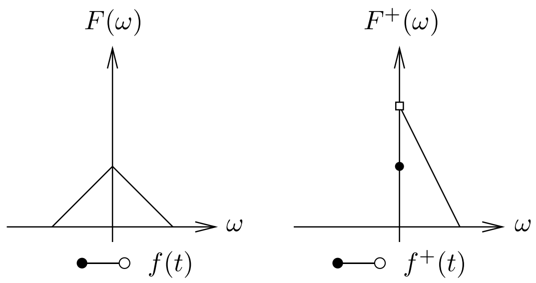
Definition 8.8 (Analytic Signal)
Let $f \in \mathbb{R} \to \mathbb{R}$ be a real signal with Fourier Transform $F(\omega)$. Let

$$\begin{aligned} F^+(\omega) &= \begin{cases} 2F(\omega) & \text{for } \omega > 0 \\ F(\omega) & \text{for } \omega = 0 \\ 0 & \text{for } \omega < 0. \end{cases} \\ &= 2\sigma(\omega)F(\omega).^a \end{aligned}$$

The analytic signal $f^+(t)$ is defined as

$$f^+(t) \quad \circ\!\!-\!\!\bullet \quad F^+(\omega).$$

[a] We use the modified step function, whose function value is 1/2 at $\omega = 0$, see footnote on page 28



Note that even though f is real, the analytic signal f^+ is complex. The analytic signal can be expressed using the Hilbert transform $\hat{f}$, see Definition 8.1.

Theorem 8.9
Let $f \in \mathbb{R} \to \mathbb{R}$ with Hilbert transform $\hat{f}$ and analytic signal f^+. Then

$$f^+(t) \quad = \quad f(t) + j\hat{f}(t).$$

Proof. Inverse Fourier transform of $F^+(\omega)$ using

$$\sigma(\omega) \quad \bullet\!\!-\!\!\circ \quad \frac{1}{2}\left(\delta(t) + \frac{j}{\pi t}\right)$$

and the Convolution Theorem give

$$f^+(t) \quad = \quad \left(\delta(t) + \frac{j}{\pi t}\right) * f(t) \quad = \quad f(t) + j\hat{f}(t). \ \square$$

It is obvious that $F(\omega)$ can be obtained from $F^+(\omega)$ as f is real. The next theorem shows how $f(t)$ can be obtained from $f^+(t)$ without Fourier transform.

Theorem 8.10

Let $f \in \mathbb{R} \to \mathbb{R}$ with analytic signal f^+. Then

$$f(t) \quad = \quad \mathrm{re}(f^+(t)).$$

Proof. As f is real, it holds that

$$F(-\omega) \quad = \quad \overline{F(\omega)}.$$

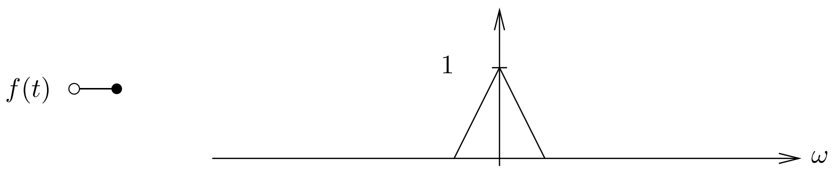
Therefore, according to Definition 8.1 and Theorem 1.11 it holds that

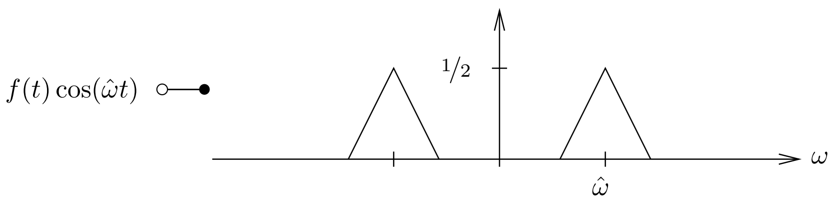
$$\begin{aligned} F(\omega) \quad &= \quad \frac{1}{2}\left(F^+(\omega) + \overline{F^+(-\omega)}\right) \\ &\bullet\!\!-\!\!\circ \quad \frac{1}{2}\left(f^+(t) + \overline{f^+(t)}\right) \\ &= \quad \mathrm{re}(f^+(t)). \ \square \end{aligned}$$

8.4 Single Side Band Amplitude Modulation

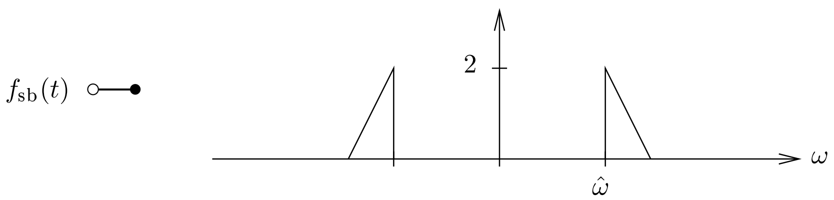
Single side band amplitude modulation is an improvement over amplitude modulation which has the advantage that it needs only half the bandwidth. It can best be understood in the frequency domain using pictures.

Let $f(t)$ be a real, band limited signal. With normal amplitude modulation we would multiply $f(t)$ with $\cos(\hat{\omega}t)$, see Section 8

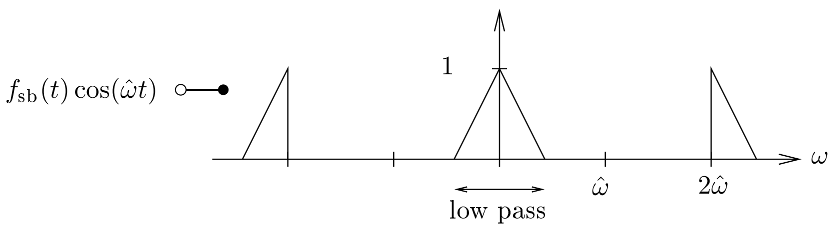




As $f(t)\cos(\hat{\omega}t)$ covers twice as much frequency space as $f(t)$, transmitting this signal is a waste of bandwidth. What we would prefer to transmit is a signal $f_{\text{sb}}(t)$, which looks like this.

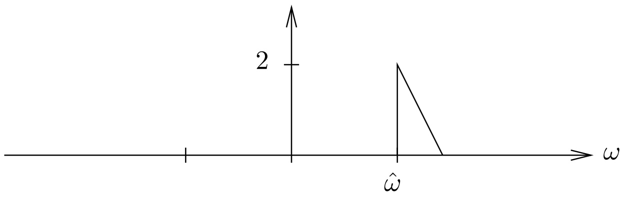


Note that f_{sb} is a real signal. Demodulating and low pass filtering $f_{\text{sb}}(t)$ at the receivers side will give us back the original $f(t)$.



The straight forward way to generate $f_{\text{sb}}(t)$ is to high pass filter $f(t)\cos(\hat{\omega}t)$ with cut off frequency $\hat{\omega}$. However, in order to avoid distortions of the signal, this would require very high edge steepness. A better way is as follows. The analytic signal $f^{+}_{\text{sb}}(t)$ of $f_{\text{sb}}(t)$ is obtained by deleting negative frequencies:

$f_{\mathrm{sb}}^{+}(t) \quad \circ\!\!-\!\!\bullet$



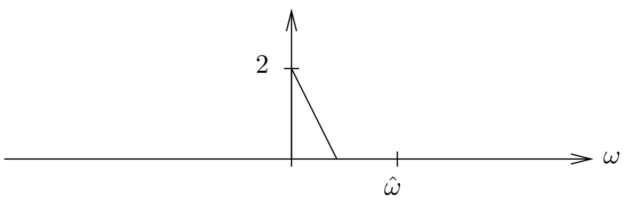
If we had $f_{\mathrm{sb}}^{+}(t)$ we could easily get $f_{\mathrm{sb}}(t)$ by taking its real part, see Theorem 8.10:

$$f_{\mathrm{sb}}(t) \quad = \quad \mathrm{re}\big(f_{\mathrm{sb}}^{+}(t)\big).$$

The Signal $f_{\mathrm{sb}}^{+}(t)$ can be obtained from $f^{+}(t)$ by shifting in frequency domanin.

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \hat{\omega})$$

$f^{+}(t) \quad \circ\!\!-\!\!\bullet$



Hence

$$f_{\mathrm{sb}}^{+}(t) \quad = \quad f^{+}(t)e^{j\hat{\omega}t}.$$

And finally the analytic signal $f^{+}(t)$ can be obtained with the Hilbert transform $\hat{f}(t)$ of $f(t)$, see Theorem 8.9

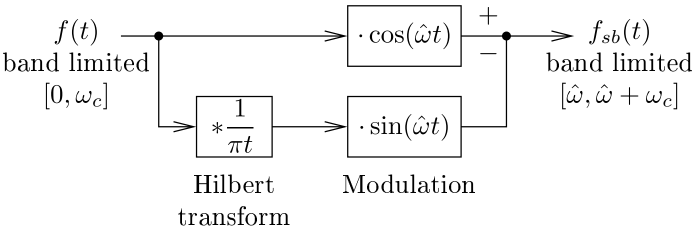
$$f^{+}(t) \quad = \quad f(t) + j\hat{f}(t).$$

Putting all steps together we have

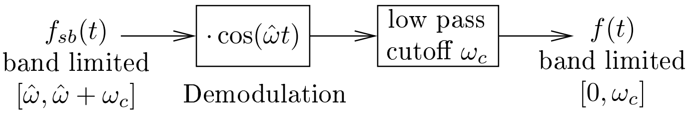
$$\begin{aligned}
f_{\mathrm{sb}}(t) &= \mathrm{re}\big(f_{\mathrm{sb}}^{+}(t)\big) \\
&= \mathrm{re}\big(f^{+}(t)e^{j\hat{\omega}t}\big) \\
&= \mathrm{re}\Big((f(t) + j\hat{f}(t))e^{j\hat{\omega}t}\Big) \\
&= f(t)\cos(\hat{\omega}t) - \hat{f}(t)\sin(\hat{\omega}t).
\end{aligned}$$

As shown in Section 8.2 the Hilbert transform $\hat{f}(t)$ can be computed easily with digital signal processing. The derivation of $f_{\mathrm{sb}}(t)$ was mostly done in the frequency domain, but the actual computation of this signal is done entirely in time domain. The result can be interpreted as a mixture of a cos- and sin-modulated signal, where a phase shift by $\pi/2$ was applied before sin-modulation. In that sense we see similarity to quadrature amplitude modulation in Section 8.1

Sender



Receiver



A Appendix

A.1 Properties of Convolution

- **Commutative Law**

$$f * g = g * f$$

- **Associative Law**

$$(f * g) * h = f * (g * h)$$

- **Linearity (including Distributive Law)**

$$\begin{aligned} (f_1 + f_2) * g &= f_1 * g + f_2 * g \\ (af) * g &= a(f * g) \end{aligned}$$

- **Time Invariance**

$$f_{\hat{t}} * g = (f * g)_{\hat{t}} = f * g_{\hat{t}}$$

- **Permutability of Convolution and Derivative**

$$f' * g = (f * g)' = f * g'$$

- **Integration by Convolution with Step Function**

$$(f * \sigma)(t) = \int_{-\infty}^{t} f(\tau) d\tau.$$

- **Neutral Element**

$$\begin{aligned} f * \delta &= f \\ f * \delta_{\hat{t}} &= f_{\hat{t}} \end{aligned}$$

- **Generalized Derivative**

$$\sigma'(t) = \delta(t)$$

- **Sifting Property**

$$\begin{aligned} f(t)\delta(t-a) &= f(a)\delta(t-a) \\ \int_{-\infty}^{\infty} f(t)\delta(t-a) &= f(a) \end{aligned}$$

A.2 Fourier Transform Pairs

$$\delta(t) \quad \circ\!\!-\!\!\bullet \quad 1$$

$$1 \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega)$$

$$e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\delta(\omega-\hat{\omega})$$

$$\sigma(t)e^{at} \quad \circ\!\!-\!\!\bullet \quad \frac{1}{j\omega-a} \qquad \text{falls } a<0$$

$$\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \pi(\delta(\omega-\hat{\omega})+\delta(\omega+\hat{\omega}))$$

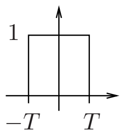
$$\sin(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad -j\pi(\delta(\omega-\hat{\omega})-\delta(\omega+\hat{\omega}))$$

$$\mathrm{sign}(t) \quad \circ\!\!-\!\!\bullet \quad \frac{2}{j\omega}$$

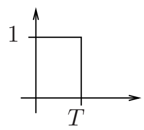
$$\frac{1}{t} \quad \circ\!\!-\!\!\bullet \quad -j\pi\mathrm{sign}(\omega)$$

$$\sigma(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{j\omega}+\pi\delta(\omega)$$

$$\frac{j}{\pi t}+\delta(t) \quad \circ\!\!-\!\!\bullet \quad 2\sigma(\omega)$$

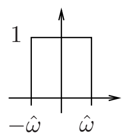


$$\circ\!\!-\!\!\bullet \quad 2T\mathrm{si}(\omega T)$$



$$\circ\!\!-\!\!\bullet \quad \begin{cases} \frac{1}{j\omega}(1-e^{-j\omega T}) & \text{falls } \omega \neq 0 \\ T & \text{falls } \omega = 0 \end{cases}$$

$$\frac{\hat{\omega}}{\pi}\mathrm{si}(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet$$



$$T_s\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{k=-\infty}^{\infty}e^{jk\omega_s t} \quad \circ\!\!-\!\!\bullet \quad 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s), \quad \omega_s=\frac{2\pi}{T_s}$$

A.3 Properties of Fourier Transform

Symmetry

$$\begin{aligned} f(t) \text{ real} \quad &\circ\!\!-\!\!\bullet \quad F(-\omega) = \overline{F(\omega)} \\ f(t) \text{ real, even} \quad &\circ\!\!-\!\!\bullet \quad F(\omega) \text{ real, even} \\ f(t) \text{ real, odd} \quad &\circ\!\!-\!\!\bullet \quad F(\omega) \text{ imaginary, odd} \end{aligned}$$

Linearity

$$\begin{aligned} f(t) + g(t) \quad &\circ\!\!-\!\!\bullet \quad F(\omega) + G(\omega) \\ af(t) \quad &\circ\!\!-\!\!\bullet \quad aF(\omega) \end{aligned}$$

Time Shift

$$f(t - \hat{t}) \quad \circ\!\!-\!\!\bullet \quad e^{-j\omega\hat{t}} F(\omega)$$

Frequency Shift

$$f(t)e^{j\hat{\omega}t} \quad \circ\!\!-\!\!\bullet \quad F(\omega - \hat{\omega})$$

Modulation

$$f(t)\cos(\hat{\omega}t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2}(F(\omega - \hat{\omega}) + F(\omega + \hat{\omega}))$$

Time Reverse

$$f(-t) \quad \circ\!\!-\!\!\bullet \quad F(-\omega) = \overline{F(\omega)}$$

Time Expansion

$$f(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Derivative in Time Domain

$$\begin{aligned} f'(t) \quad &\circ\!\!-\!\!\bullet \quad (j\omega)F(\omega) \\ f''(t) \quad &\circ\!\!-\!\!\bullet \quad (j\omega)^2 F(\omega) \end{aligned}$$

Integration in Time Domain

$$\int_{-\infty}^{t} f(u)du \quad \circ\!\!-\!\!\bullet \quad \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) F(\omega)$$

Derivative in Frequency Domain

$$\begin{aligned} (-jt)f(t) \quad &\circ\!\!-\!\!\bullet \quad F'(\omega) \\ (-jt)^2 f(t) \quad &\circ\!\!-\!\!\bullet \quad F''(\omega) \end{aligned}$$

Convolution in Time Domain

$$(f * g)(t) \quad \circ\!\!-\!\!\bullet \quad F(\omega)G(\omega)$$

Convolution in Frequency Domain

$$f(t)g(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{2\pi}(F * G)(\omega)$$

Sampling

$$\underbrace{f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)}_{\text{Sampling}} = \sum_{n=-\infty}^{\infty} f_n\delta(t-nT_s) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{T_s}\underbrace{\sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)}_{\text{Periodic Continuation}}$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

References

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